

A Nonabelian Yang-Mills Analogue of Classical Electromagnetic Duality

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Abstract

The classic question of a nonabelian Yang-Mills analogue to electromagnetic duality is here examined in a minimalist fashion at the strictly 4-dimensional, classical field and point charge level. A generalisation of the abelian Hodge star duality is found which, though not yet known to give dual symmetry, reproduces analogues to many dual properties of the abelian theory. For example, there is a dual potential, but it is a 2-indexed tensor $T_{\mu\nu}$ of the Freedman-Townsend type. Though not itself functioning as such, $T_{\mu\nu}$ gives rise to a dual parallel transport, \tilde{A}_μ , for the phase of the wave function of the colour magnetic charge, this last being a monopole of the Yang-Mills field but a source of the dual field. The standard colour (electric) charge itself is found to be a monopole of \tilde{A}_μ . At the same time, the gauge symmetry is found doubled from say $SU(N)$ to $SU(N) \times SU(N)$. A novel feature is that all equations of motion, including the standard Yang-Mills and Wong equations, are here derived from a ‘universal’ principle, namely the Wu-Yang (1976) criterion for monopoles, where interactions arise purely as a consequence of the topological definition of the monopole charge. The technique used is the loop space formulation of Polyakov (1980).

1 Introduction

Recently, questions of electromagnetic duality have again come to the fore. Apart from the related duality in superstring theory [1]-[4], there have been much interest in dual symmetric actions for field theories [5] and of course the recent work on electromagnetic duality in supersymmetric theories of Seiberg, Witten and co-workers [6]-[8]. Although these recent works are based on sophisticated solitonic structures in sophisticated theories, they originate nevertheless from the simple property of the Maxwell theory of being invariant under the interchange of electricity and magnetism, which though long known, has retained a continued and recurrent interest among physicists.

The reason is that dual symmetry in electromagnetism has associated with it many important consequences. For example, the fact that the Maxwell field $F_{\mu\nu}$ is a gauge field derivable from a gauge potential A_μ implies that the dual field $*F_{\mu\nu}$ must also be derivable from some potential \tilde{A}_μ . And just as A_μ acts as the parallel phase transport for the wave functions of electric charges, so its dual partner \tilde{A}_μ will act as the parallel phase transport for the wave functions of particles carrying a magnetic charge. Hence, since the theory is invariant under a $U(1)$ gauge transformation: $A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$, it follows that it must also be invariant under a $\tilde{U}(1)$ transformation: $\tilde{A}_\mu \longrightarrow \tilde{A}_\mu + \partial_\mu \tilde{\Lambda}$, giving in all a $U(1) \times \tilde{U}(1)$ invariance. Further, although it is conventional to regard an electric charge as a source and a magnetic charge as a monopole of the Maxwell field $F_{\mu\nu}$, it is equally admissible, because of dual symmetry, to consider an electric charge as a monopole and a magnetic charge as a source of the dual Maxwell field $*F_{\mu\nu}$.¹ Thus, it is possible to apply the Wu-Yang criterion [9] for monopoles to derive, for example, the Lorentz equation for either an electric or a magnetic charge from the monopole's definition as a topological obstruction without explicitly introducing an interaction term into the action. Indeed, in Charts I and II below, we have listed a whole complex of properties of electromagnetism, some perhaps less well-known than the above-mentioned, which are all connected in some way with the concept of duality.

Given the importance of nonabelian Yang-Mills theories in present day physics, it is natural to ask whether the notion of electromagnetic duality extends to them as well. In the recent developments, this question is addressed in a broad daring fashion by Seiberg, Witten and co-workers, in which electromagnetic duality is embedded in a grand unified (at present supersymmetric) quantum field framework, where charges (whether electric or magnetic) appear as 't Hooft-Polyakov solitons. They not only give strong evidence for an exact duality in supersymmetric Yang-Mills theories as suggested by Olive and Montonen [10] but trigger also a breakthrough in 4-manifold topology. (See [8] and recent work of C. Taubes on Donaldson invariants.) More work has also been done to support and generalise

¹Here, a source is considered to give rise to a current represented by the divergence of the field, while a monopole results from a topologically nontrivial configuration of the potential.

the above [11] The result is an imposing structure with rich and exciting properties which is being diligently explored by both the physics and mathematics communities.

Our present work addresses the same basic question but starts from a complementary minimalist standpoint. The idea is that since the attractive dual properties listed above occur for electromagnetism already at the classical field level with point charges, it makes sense also to ask whether (nonabelian) Yang-Mills theories may share some of these properties already at the same classical field and point charge level.

In terms of the Maxwell fields $F_{\mu\nu}$, symmetry under the interchange of electricity and magnetism, i.e. $\mathbf{E} \rightarrow -\mathbf{H}, \mathbf{H} \rightarrow \mathbf{E}$, means an invariance under the Hodge star operation: $F_{\mu\nu} \rightarrow {}^*F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. In Yang-Mills theory, on the other hand, although the action is still invariant (apart from a sign) under $F_{\mu\nu} \rightarrow {}^*F_{\mu\nu}$, the dual field ${}^*F_{\mu\nu}$, in contrast to $F_{\mu\nu}$, is not in general a gauge field derivable from a potential. [12] Hence, if duality is still taken to mean the Hodge star operation $F_{\mu\nu} \rightarrow {}^*F_{\mu\nu}$ for the nonabelian theory, then strict dual symmetry is lost. However, it is possible to entertain the notion of a generalised duality relationship in Yang-Mills theory, which, though reducing to the Hodge star operation in the abelian theory, is not identical to it when the theory is nonabelian. The only question is whether such a notion would be profitable. Considerations can then be taken at two levels. One notes that the ‘dual’ properties of electromagnetism listed in Charts I and II do not by themselves imply dual symmetry although they are consequences of it. Thus a generalised duality relationship may be such as to reproduce all the dual properties of Charts I and II for Yang-Mills fields without restoring dual symmetry and it would still be a very useful concept.

What we wish to exhibit below is a generalised duality relationship in Yang-Mills fields, which though not known as yet to give a fully-fledged dual symmetry manages nevertheless to reproduce nonabelian analogues to all the dual properties of electromagnetism listed in Charts I and II. For example, one finds that there is in nonabelian theory indeed a ‘generalised dual potential’, whose ‘curl’ or ‘exterior derivative’, though not ${}^*F_{\mu\nu}$, is a quantity closely related to it. This ‘dual potential’ is not a vector quantity but an antisymmetric second-rank tensor $T_{\mu\nu}$ of the type first found in superstring theory [13], but although it does not itself act as the parallel phase transport for wave functions of the colour magnetic charge, it gives rise to another quantity \tilde{A}_μ which plays that role. In terms of $T_{\mu\nu}$, the Yang-Mills action takes on a Freedman-Townsend form [14] invariant under the transformation $T_{\mu\nu} \longrightarrow T_{\mu\nu} + \delta_{[\mu}\Lambda_{\nu]}$, which induces in turn in \tilde{A}_μ a new $SU(N)$ transformation of odd parity, giving thus, as in the abelian theory, a doubling of the original gauge symmetry, from $SU(N)$ to $SU(N) \times \widetilde{SU(N)}$. Further, it will appear that the sources of one potential can again be regarded as monopoles of its ‘dual’ and vice versa, so that, if we call a source of the Yang-Mills potential A_μ a ‘colour electric’ and a monopole of A_μ a ‘colour magnetic’ charge, then a ‘colour electric’ charge can also be regarded as a ‘monopole’ and a ‘colour mag-

netic' charge a source of the 'dual potential' $T_{\mu\nu}$. It even follows then that both the Yang-Mills equation and its classical limit, the Wong equation [15], can be derived from the definition of a monopole using the above-mentioned Wu-Yang criterion. The full result is summarised in Charts III and IV below, which are seen to parallel closely Charts I and II of the abelian theory.

We note that in spite of this close analogy to the abelian theory, the fact remains that the nonabelian theory is not yet known to be fully symmetric. Thus although both 'colour electric' and 'colour magnetic' charges can be considered as monopoles in some field so that the Wu-Yang criterion can be used in each case to derive the equations of motion, the two 'dual' sets of equations so obtained need no longer describe the same dynamics as they did in the abelian theory. For the 'colour electric' charge, the equations turn out to be exactly those of the standard Yang-Mills theory, in spite of their very different derivation from the conventional. The equations for the 'colour magnetic' charge, on the other hand, may describe a different dynamics which is only just beginning to be explored. [16]

In exhibiting the analogue duality of Yang-Mills theory, we have used the loop space description of gauge theory in terms of some loop variables first introduced by Polyakov in 1980, [17]. Although we are unsure whether this apparatus is essential, we believe it to be appropriate for the following reason. The question of duality is closely connected to the existence or otherwise of a 'dual potential', which is in turn intimately related to the presence or absence of monopoles. The virtue of the Polyakov loop variables is that, unlike the standard local variables A_μ and $F_{\mu\nu}$, they remain patch-independent in the Wu-Yang [18] sense (or nonsingular in the sense of not having any Dirac strings) even in the presence of monopoles. They are thus particularly suited to the problem.

It should be stressed, however, that in spite of this loop formulation, there has been no input other than the standard Yang-Mills theory in deriving the results, and that we have worked throughout in a space-time of strictly 4 dimensions. The dual properties we have found are thus of ordinary Yang-Mills fields, only cast in a somewhat unconventional language.

In order to exhibit fully the nonabelian dual structure under consideration, much effort in this paper has to be spent in reformulating and reorganising information of results contained in our earlier work. The main new specific results of this paper are Sections 4 and 6, of which the most interesting we think is in Section 6 where it is shown that ordinary colour (electric) charges are monopoles of the dual formulation and that their known dynamics can be derived from the Wu-Yang criterion for monopoles.

2 Duality in Electromagnetism

Let us begin by recalling some of the known dual properties of electromagnetism so as to facilitate comparison with the new results we shall later derive for nonabelian

theories. [19, 20, 21]

Consider first pure electrodynamics, with neither electric nor magnetic charges in the theory. One starts then usually with a gauge field $F_{\mu\nu}(x)$ derivable from a potential $A_\mu(x)$, thus:

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x). \quad (2.1)$$

The field action is taken to be:²

$$\mathcal{A}_F^0 = -\frac{1}{16\pi} \int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x), \quad (2.2)$$

where $F_{\mu\nu}(x)$ is given in terms of $A_\mu(x)$ through (2.1), and equations of motion are to be obtained by extremising \mathcal{A}_F^0 with respect to $A_\mu(x)$, giving:

$$\partial^\nu F_{\mu\nu}(x) = 0. \quad (2.3)$$

Define next the ‘dual field’ $*F_{\mu\nu}(x)$ as:

$$*F_{\mu\nu}(x) = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}(x), \quad (2.4)$$

where the Hodge star operation $*$ exchanges electricity and magnetism thus: $\mathbf{E} \rightarrow -\mathbf{H}$, $\mathbf{H} \rightarrow \mathbf{E}$. In terms of $*F_{\mu\nu}(x)$, the equation (2.3) is equivalent to the Bianchi identity:

$$\partial_\lambda *F_{\mu\nu}(x) + \partial_\mu *F_{\nu\lambda}(x) + \partial_\nu *F_{\lambda\mu}(x) = 0, \quad (2.5)$$

which implies, in this abelian case, via the Poincaré lemma in 4-dimensional space-time that $*F_{\mu\nu}(x)$ must also be a gauge field locally derivable from some potential $\tilde{A}_\mu(x)$, thus:

$$*F_{\mu\nu}(x) = \partial_\nu \tilde{A}_\mu(x) - \partial_\mu \tilde{A}_\nu(x). \quad (2.6)$$

Hence one sees that starting with a gauge field $F_{\mu\nu}(x)$ satisfying (2.1), one ends up via the the equation of motion (2.3) with an $*F_{\mu\nu}(x)$ which is also a gauge field derivable from a potential. This is an essential feature of electromagnetic duality.

Further, it follows that in addition to the original gauge invariance under the transformation:

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \Lambda(x), \quad (2.7)$$

there is a gauge invariance also under the transformation:

$$\tilde{A}_\mu(x) \longrightarrow \tilde{A}_\mu(x) + \partial_\mu \tilde{\Lambda}(x), \quad (2.8)$$

giving the theory in all a $U(1) \times U(1)$ gauge symmetry. The second $U(1)$ has in fact the opposite parity to the first because of the Hodge star operation and will henceforth be referred to as the \tilde{U} -invariance to distinguish it from the first $U(1)$. Note, however, that since $\tilde{A}_\mu(x)$ is related to $A_\mu(x)$ through (2.1), (2.4), and (2.6),

²In our convention, $g_{\mu\nu} = (+, -, -, -)$, $\epsilon_{0123} = 1$.

the two $U(1)$ symmetries do not correspond to two distinct physical degrees of freedom.

There is an alternative method for treating the above problem which, though yielding nothing new in pure electrodynamics, is particularly amenable for future adaptation both to electrodynamics with electric or magnetic charges and to non-abelian Yang-Mills theories. [19, 20, 21] Instead of the potential $A_\mu(x)$, one adopts $F_{\mu\nu}(x)$ as field variables, but under the constraint:

$$\partial_\lambda F_{\mu\nu}(x) + \partial_\mu F_{\nu\lambda}(x) + \partial_\nu F_{\lambda\mu}(x) = 0, \quad (2.9)$$

or equivalently:

$$\partial^\nu {}^*F_{\mu\nu}(x) = 0. \quad (2.10)$$

The set of variables $\{F_{\mu\nu}(x)\}$ is of course inherently redundant, comprising more components than the original set $\{A_\mu(x)\}$ which we know is already sufficient for describing our system. However, by imposing the constraint (2.9) or (2.10), this redundancy is removed since (2.9) is the Bianchi identity for $F_{\mu\nu}(x)$ and hence, as above for the case of ${}^*F_{\mu\nu}(x)$, ensures via the Poincaré lemma that $F_{\mu\nu}(x)$ is a gauge field derivable locally from a potential as in (2.1). Hence the problem as formulated now in terms of $F_{\mu\nu}(x)$ is the same as the original problem formulated in terms of $A_\mu(x)$. Equations of motion are now to be obtained by extremising \mathcal{A}_F^0 in (2.2) with respect to $F_{\mu\nu}(x)$ but under the constraint (2.10). Introducing then Lagrange multipliers $\lambda_\mu(x)$ and forming the action:

$$\mathcal{A}_F = -\frac{1}{16\pi} \int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x) + \int d^4x \lambda_\mu(x) \partial_\nu {}^*F^{\mu\nu}(x), \quad (2.11)$$

we then obtain the equation:

$$F_{\mu\nu}(x) = 4\pi \left\{ \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} [\partial^\sigma \lambda^\rho(x) - \partial^\rho \lambda^\sigma(x)] \right\}, \quad (2.12)$$

or:

$${}^*F_{\mu\nu}(x) = 4\pi [\partial_\nu \lambda_\mu(x) - \partial_\mu \lambda_\nu(x)], \quad (2.13)$$

which states that ${}^*F_{\mu\nu}(x)$ is a gauge field with 4π times the Lagrange multiplier $\lambda_\mu(x)$ as the gauge potential:

$$\tilde{A}_\mu(x) = 4\pi \lambda_\mu(x). \quad (2.14)$$

In turn, (2.13) implies that ${}^*F_{\mu\nu}(x)$ must satisfy the Bianchi identity (2.5), or equivalently the Maxwell equation (2.3). In other words, one obtains just exactly the same result as before, as expected. However, this alternative treatment has two new features, which will facilitate the analysis of the duality question. First, it is formulated in terms of gauge invariant variables $F_{\mu\nu}(x)$ which, as we shall see later, are particularly useful in solving the parallel problem in the presence of monopoles. Secondly, the dual potential appears automatically as the Lagrange multiplier to the constraint imposed on these variables to remove their inherent

redundancy, which will afford later a valuable hint on how the dual potential may be generalised to the nonabelian theory.

From (2.4), one sees that:

$$*(F_{\mu\nu}(x)) = -F_{\mu\nu}(x), \quad (2.15)$$

meaning that (apart from a sign) $F_{\mu\nu}(x)$ is also the dual of $*F_{\mu\nu}(x)$, and that when expressed in terms of $*F_{\mu\nu}(x)$, \mathcal{A}_F^0 in (2.2) is of exactly the same form (also apart from a sign):

$$\tilde{\mathcal{A}}_F^0 = \frac{1}{16\pi} \int d^4x *F_{\mu\nu}(x) *F^{\mu\nu}(x). \quad (2.16)$$

From these facts, it follows that all our preceding arguments would remain valid under the interchange of $F_{\mu\nu}(x)$ with $*F_{\mu\nu}(x)$ and $A_\mu(x)$ with $\tilde{A}_\mu(x)$, which is what one usually calls dual symmetry.

For easy reference in the future, the dual structure of pure electrodynamics as elucidated above is summarised in the flowchart Chart I.

Next, consider electrodynamics in the presence of electric or magnetic charges. Conventionally, one is used to regarding electric charges as ‘sources’ and magnetic charges as ‘monopoles’ of the Maxwell field $F_{\mu\nu}(x)$ in the sense that electric charges give nonzero divergences to the field while magnetic charges result from nontrivial topological configurations of the potential. Because of the dual symmetry described above, however, it is equally valid to regard electric charges as ‘monopoles’ and magnetic charges as ‘sources’ of the dual Maxwell field $*F_{\mu\nu}(x)$.

In formulating the interaction of an electric or a magnetic charge with the electromagnetic field, the usual approach is to regard the charge as a source. Thus, for a classical point electric charge e , one chooses usually to work with the Maxwell field $F_{\mu\nu}(x)$ in which e appears as a source, and write for the action:

$$\mathcal{A}^0 = -\frac{1}{16\pi} \int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x) - m \int d\tau. \quad (2.17)$$

Or else, if the charge e is to be carried by a quantum particle described by, say, a Dirac wave function $\psi(x)$, we write:

$$\mathcal{A}^0 = -\frac{1}{16\pi} \int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x) + \int d^4x \bar{\psi}(x) (i\partial_\mu \gamma^\mu - m) \psi(x). \quad (2.18)$$

The interaction between the charge and the field is then introduced as an additional term in the action, thus:

$$\mathcal{A}_I = e \int A_\mu(x) dx^\mu, \quad (2.19)$$

for the classical charge, where the integral is to be taken along the world-line $Y^\mu(\tau)$ of the particle, or else:

$$\mathcal{A}_I = e \int d^4x \bar{\psi}(x) A_\mu(x) \gamma^\mu \psi(x), \quad (2.20)$$

for the quantum charge.

The alternative approach given above for pure electromagnetism was originally formulated for charges treated as monopoles rather than as sources, in which case their interaction appears as a unique consequence of the topological definition of a monopole. This is contained in a paper of Wu and Yang in 1976 and will be referred to here henceforth as the Wu-Yang criterion. [9, 19, 20, 21] Intuitively, the idea is as follows. Monopoles are topological obstructions in gauge fields, because of which their charges are necessarily quantised and conserved. That there exists a monopole at some point in space means that the gauge field has a certain topological configuration around that point. Thus, if the monopole moves to another position, the gauge field will have to rearrange itself so as to maintain the same topological configuration around the new point. Hence, the definition of a monopole as a topological obstruction already contains within it an implied coupling between the monopole's position and the surrounding field, or in physical terms a charge-field interaction. The only question is whether this intrinsic interaction is the same as that of the usual formulation.

To answer this question, we first recall briefly the description of a monopole as a topological obstruction. [20] Since electrodynamics is dual symmetric, we can work either with an electric charge regarded as a monopole of $*F_{\mu\nu}(x)$ or with a magnetic charge regarded as a monopole of $F_{\mu\nu}(x)$. We choose here to illustrate with the latter since the notation there is more familiar. Take a one-parameter family of closed loops in space enveloping a closed surface Σ . Label the loops as $\{C_t\}$ with the parameter t ranging in value from 0 to 2π so that at $t = 0$ and $t = 2\pi$ C_t shrinks to the point P_0 . For each of these loops, one can define a phase factor:

$$\Phi(C) = \exp ie \oint_C A_\mu(x) dx^\mu, \quad (2.21)$$

which is an element of the gauge group $U(1)$. As t varies from 0 to 2π , the image of C_t by Φ in $U(1)$ will trace a closed curve, say Γ_Σ , in $U(1)$, beginning and ending at the group identity. The gauge group $U(1)$ here has the topology of a circle S^1 so that the closed curve Γ_Σ must wind around this circle an integral number of times. In other words, the total change in phase of $\Phi(C_t)$ for $t = 0 \rightarrow 2\pi$ must equal $2\pi n$, with n integral. This value of n , being discrete, cannot be changed by any continuous deformation of the surface Σ . Hence, for $n \neq 0$, the gauge field will behave as if there is some object enclosed inside Σ which forbids Σ from trivially shrinking to zero. This is what is termed a topological obstruction. Further, it is easily seen that the total change in phase of $\Phi(C_t)$ for $t = 0 \rightarrow 2\pi$ is just the total magnetic flux flowing out of the surface Σ , or else 4π times the magnetic charge enclosed inside Σ . Hence, we conclude that the topological obstruction here represents a magnetic monopole with a quantised charge \tilde{e} .

To derive the equations of motion of a classical magnetic charge interacting with an electromagnetic field, our considerations above require that we extremise the action (2.17) subject to the condition that the particle $Y^\mu(\tau)$ represents a topological obstruction of the type described in the preceding paragraph. It can

readily be shown that such a condition can be written in the following Lorentz covariant form:

$$\partial_\nu {}^*F^{\mu\nu}(x) = -4\pi\tilde{e} \int d\tau \frac{dY^\mu(\tau)}{d\tau} \delta(x - Y(\tau)). \quad (2.22)$$

In solving the variational problem for the equations of motion, one can in principle retain the gauge potential $A_\mu(x)$ as variables but since, as noted above, $A_\mu(x)$ has to be patched in the presence of the monopole, this would make the problem rather unwieldy, especially since the patches will have to depend on the position Y of the monopole. This suggests that one adopts instead the alternative approach using $F_{\mu\nu}(x)$ as variables but, in that case, one would have to constrain $F_{\mu\nu}(x)$ so as to ensure that their inherent redundancy be removed or that the potential $A_\mu(x)$ can be recovered from them. The beauty, however, is that the constraint we need for removing the redundancy is already exactly contained in the condition (2.22) we wish to impose. Indeed, (2.22) implies that, except on the monopole world-line $Y^\mu(\tau)$, the Bianchi identity (2.10) or (2.9) is satisfied, so that by the Poincaré lemma $F_{\mu\nu}(x)$ is derivable from a potential, whereas on $Y^\mu(\tau)$ itself, we do not expect to have a potential $A_\mu(x)$ in any case. Thus, in analogy to the previous treatment of pure electrodynamics, one can just use $F_{\mu\nu}(x)$ as field variables in place of $A_\mu(x)$ so long as (2.22) is satisfied, and so avoid all problems with patching.

To derive the equations of motion then, we extremise the free action (2.17) with respect to $F_{\mu\nu}(x)$ and $Y^\mu(\tau)$ under the constraint (2.22). Introducing Lagrange multipliers $\lambda_\mu(x)$ as before, we extremise instead:

$$\mathcal{A} = \mathcal{A}^0 + \int d^4x \lambda_\mu(x) \{ \partial_\nu {}^*F^{\mu\nu}(x) + 4\pi\tilde{e} \int d\tau \frac{dY^\mu(\tau)}{d\tau} \delta(x - Y(\tau)) \}. \quad (2.23)$$

The solution is straightforward, giving again (2.13) as well as:

$$m \frac{d^2 Y_\mu(\tau)}{d\tau^2} = -4\pi\tilde{e} \{ \partial_\nu \lambda_\mu(Y(\tau)) - \partial_\mu \lambda_\nu(Y(\tau)) \} \frac{dY^\nu(\tau)}{d\tau}, \quad (2.24)$$

which, on eliminating $\lambda_\mu(x)$ by (2.13), yields respectively (2.3) and:

$$m \frac{d^2 Y^\mu(\tau)}{d\tau^2} = -\tilde{e} {}^*F^{\mu\nu}(Y(\tau)) \frac{dY_\nu(\tau)}{d\tau}. \quad (2.25)$$

Together with the original constraint (2.22), (2.3) and (2.25) then constitute the complete set of equations of motion for the magnetic charge interacting with the electromagnetic field.

Since the system is dual symmetric, exactly the same arguments would apply for an electric charge when considered as a monopole of the dual Maxwell field ${}^*F_{\mu\nu}(x)$. The constraint defining the charge as a topological obstruction would then be:

$$\partial_\nu F^{\mu\nu}(x) = -4\pi e \int d\tau \frac{dY^\mu(\tau)}{d\tau} \delta(x - Y(\tau)), \quad (2.26)$$

under which the action (2.17), or equivalently the action in terms of ${}^*F_{\mu\nu}(x)$:

$$\tilde{A}^0 = \frac{1}{16\pi} \int d^4x {}^*F_{\mu\nu}(x) {}^*F^{\mu\nu}(x) - m \int d\tau, \quad (2.27)$$

has to be extremised with respect to ${}^*F_{\mu\nu}(x)$ and $Y^\mu(\tau)$ as variables. Proceeding as above, we get:

$${}^*F_{\mu\nu}(x) = 4\pi \left\{ \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} [\partial^\sigma \tilde{\lambda}^\rho(x) - \partial^\rho \tilde{\lambda}^\sigma(x)] \right\}, \quad (2.28)$$

and:

$$m \frac{d^2 Y^\mu(\tau)}{d\tau^2} = -e F^{\mu\nu}(Y(\tau)) \frac{dY_\nu(\tau)}{d\tau}. \quad (2.29)$$

(2.28) says that $F_{\mu\nu}(x)$ can be derived from a potential as in (2.1) with:

$$A_\mu(x) = 4\pi \tilde{\lambda}_\mu(x), \quad (2.30)$$

The equations (2.26), (2.1), and (2.29) are exactly the standard Maxwell and Lorentz equations governing the motion of an electric charge moving in an electromagnetic field as normally derived by extremising:

$$\mathcal{A}' = \mathcal{A}^0 + \mathcal{A}_I \quad (2.31)$$

with respect to $A_\mu(x)$ and $Y^\mu(\tau)$ for \mathcal{A}^0 in (2.17) and \mathcal{A}_I in (2.19). The present approach is thus entirely equivalent to the conventional approach as expected, but has the virtue of giving the interaction uniquely as a consequence of the topological definition of the charge without introducing an interaction term in the action, the choice of which in the conventional approach seems to admit additional freedom.

The formulation for a quantum charged particle when considered as a monopole is similar. For a magnetic charge, we extremise then \mathcal{A}^0 in (2.18) with respect to $F_{\mu\nu}(x)$ and $\psi(x)$ under the constraint:

$$\partial_\nu {}^*F^{\mu\nu}(x) = -4\pi \tilde{e} \bar{\psi}(x) \gamma^\mu \psi(x), \quad (2.32)$$

where we have just replaced in (2.22) the classical current by the quantum mechanical current. Using again the method of Lagrange multipliers and varying:

$$\mathcal{A} = \mathcal{A}_0 + \int d^4x \lambda_\mu(x) \{ \partial_\nu {}^*F^{\mu\nu}(x) + 4\pi \tilde{e} \bar{\psi}(x) \gamma^\mu \psi(x) \}, \quad (2.33)$$

with respect to $F_{\mu\nu}(x)$ and $\psi(x)$ we obtain (2.12) and:

$$(i\partial_\mu \gamma^\mu - m)\psi(x) = -\tilde{e} \tilde{A}_\mu(x) \gamma^\mu \psi(x), \quad (2.34)$$

for $\tilde{A}_\mu(x)$ as given in (2.14), which are exactly the dual of the standard equations for an electric charge moving in a Maxwell field. Indeed, because of dual symmetry, one readily sees that had we started with an electric charge considered as a monopole in the dual Maxwell field ${}^*F_{\mu\nu}(x)$, we would have obtained the standard equations for an electric charge.

Finally, we note that as for pure electrodynamics, the systems with electric or magnetic charges will still have the gauge symmetry doubled to $U(1) \times \widetilde{U(1)}$. Take the action (2.33) for example, which will be useful later for comparison with the parallel in the nonabelian theory. Under the original $U(1)$ symmetry, all the quantities appearing in \mathcal{A} are invariant, including $\psi(x)$ which, though magnetically charged, is electrically neutral. Under a \tilde{U} -transformation, on the other hand, $*F_{\mu\nu}(x)$ is invariant, $\lambda_\mu(x)$ transforms as in (2.8) and $\psi(x)$ transforms as:

$$\psi(x) \longrightarrow (1 + i\tilde{e}\tilde{\Lambda}(x))\psi(x). \quad (2.35)$$

Hence, in (2.33), \mathcal{A}_F^0 in \mathcal{A}^0 is invariant, and so is the first term in the integral as can be seen by an integration by parts using the antisymmetry of $*F_{\mu\nu}(x)$, while the variation from the second term in the integral will cancel with the increment from the free action of ψ in (2.18). Thus the symmetry is indeed doubled as claimed.

For easier comparison with later extensions to nonabelian theories, the dual structure as revealed in the above treatment is summarised in Chart II.

3 Pure Yang-Mills Theory - Direct Formulation

In Yang-Mills theory, one usually starts also with a gauge field $F_{\mu\nu}(x)$ derivable from a potential, thus:³

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + ig[A_\mu(x), A_\nu(x)]. \quad (3.1)$$

Equations of motion are obtained by extremising the action:

$$\mathcal{A}_F^0 = -\frac{1}{16\pi} \int d^4x \operatorname{Tr}\{F_{\mu\nu}(x)F^{\mu\nu}(x)\}, \quad (3.2)$$

with respect to $A_\mu(x)$. This gives:

$$D_\nu F^{\mu\nu}(x) = 0, \quad (3.3)$$

where D_ν is the covariant derivative defined as:

$$D_\mu = \partial_\mu - ig[A_\mu(x), \quad]. \quad (3.4)$$

So far, the analogy to electromagnetism is still rather close.

However, the result in electromagnetism that the equation of motion (2.3) implies the existence of a potential also for $*F_{\mu\nu}(x)$ no longer holds in general for nonabelian Yang-Mills theories. [12] One may still define, of course, a dual

³Although our results apply to all $su(N)$ theories, they will be given explicitly only for $su(2)$ for convenience. In our convention for $su(2)$, $B = B^i t_i$, $t_i = \tau_i/2$, $\operatorname{Tr} B = 2 \times$ sum of diagonal elements, so that $\operatorname{Tr}(t_i t_j) = \delta_{ij}$.

field $*F_{\mu\nu}(x)$ as in (2.4) for which the equation (3.3) appears again as the Bianchi identity:

$$D_\lambda *F_{\mu\nu}(x) + D_\mu *F_{\nu\lambda}(x) + D_\nu *F_{\lambda\mu}(x) = 0. \quad (3.5)$$

However, in the absence of a nonabelian analogue to the Poincaré lemma, this no longer implies the existence of a ‘dual potential’ related to $*F_{\mu\nu}(x)$ in the same way that $A_\mu(x)$ is related to $F_{\mu\nu}(x)$ in (3.1). Indeed, it has been shown explicitly by Gu and Yang [12] that for some particular examples of $*F_{\mu\nu}(x)$ satisfying (3.5) there does not exist any $\bar{A}_\mu(x)$ such that:

$$*F_{\mu\nu}(x) = \partial_\nu \bar{A}_\mu(x) - \partial_\mu \bar{A}_\nu(x) + ig[\bar{A}_\mu(x), \bar{A}_\nu(x)]. \quad (3.6)$$

The failure of (3.6) destroys any hope of a dual symmetry for nonabelian Yang-Mills theories in strict analogy to that in electromagnetism. However, there may be a possibility that by defining a different dual relationship for Yang-Mills fields which, though reducing to ordinary electromagnetic duality for the abelian theory, yet allows for the existence of a generalised dual potential. The only question is how many of the so-called ‘dual properties’ of electromagnetism are retained in this generalised dual relationship for Yang-Mills fields.

To explore such a possibility, we note first that in the alternative formulation of pure electrodynamics as summarised in the second column of Chart I, the ‘dual potential’ emerged spontaneously as the Lagrange multiplier for a constraint imposed on the variable $F_{\mu\nu}(x)$ to guarantee the existence of the ordinary potential $A_\mu(x)$. This suggests then that perhaps in nonabelian Yang-Mills theories, a generalised dual potential will also emerge as the Lagrange multiplier to a similar constraint. A possible strategy would thus be to examine how this alternative formulation of pure electrodynamics can be extended to nonabelian Yang-Mills theories.

With this view in mind, one notices immediately some important differences between the abelian and nonabelian theories. First, $F_{\mu\nu}(x)$ is gauge invariant in electromagnetism but only covariant in nonabelian Yang-Mills theory. Second, $F_{\mu\nu}(x)$ fully describes the classical abelian theory, but for the nonabelian theory, it is known that several gauge inequivalent potentials can correspond to the same $F_{\mu\nu}(x)$, meaning that $F_{\mu\nu}(x)$ is insufficient to describe fully even the classical Yang-Mills field. Third, the condition (2.10) guarantees in the abelian theory that $F_{\mu\nu}(x)$ is a gauge field derivable from a potential, but for the nonabelian theory, the corresponding condition:

$$D_\nu *F^{\mu\nu}(x) = 0, \quad (3.7)$$

cannot even be given a meaning until the potential $A_\mu(x)$ is defined. Thus, in order to mimic the above ‘alternative formulation’ of electromagnetism, one has first to find some suitable gauge invariant set of field variables which can fully describe the theory, and then to impose on them appropriate constraints so as to enable one to recover the gauge potentials.

A solution to this problem has been found using loop variables which we summarise as follows. We begin by defining *parametrised loops* as maps of the circle S^1 to 4-dimensional space-time, thus:

$$\xi : \{\xi^\mu(s); s = 0 \rightarrow 2\pi, \xi^\mu(0) = \xi^\mu(2\pi) = \xi_0^\mu\}. \quad (3.8)$$

We need to consider only those loops beginning and ending at some fixed reference point $P_0 = \{\xi_0^\mu\}$. For each parametrised loop ξ , one can then define a phase factor (Dirac factor or Wilson loop):

$$\Phi[\xi] = P_s \exp ig \int_0^{2\pi} ds A_\mu(\xi(s)) \dot{\xi}^\mu(s), \quad (3.9)$$

where P_s denotes ordering, say from right to left, and a dot differentiation in the loop parameter s . Following Polyakov [17], define next:

$$F_\mu[\xi|s] = \frac{i}{g} \Phi^{-1}[\xi] \delta_\mu(s) \Phi[\xi], \quad (3.10)$$

where we denote by $\delta_\mu(s)$ the ordinary functional derivative $\delta/\delta\xi^\mu(s)$ with respect to $\xi^\mu(s)$.⁴ This quantity $F_\mu[\xi|s]$, which may be thought of as a ‘connection’ giving parallel phase transport in loop space, is what is proposed as the field variable for Yang-Mills fields analogous to $F_{\mu\nu}(x)$ above for electromagnetism.

In terms of ordinary local field variables, $F_\mu[\xi|s]$ can be expressed as:

$$F_\mu[\xi|s] = \Phi_\xi^{-1}(s, 0) F_{\mu\nu}(\xi(s)) \Phi_\xi(s, 0) \dot{\xi}^\nu(s), \quad (3.13)$$

where:

$$\Phi_\xi(s_2, s_1) = P_s \exp ig \int_{s_1}^{s_2} ds A_\mu(\xi(s)) \dot{\xi}^\mu(s), \quad (3.14)$$

is the parallel phase transport from s_1 to s_2 . From (3.13) one sees that $F_\mu[\xi|s]$ is closely related to the local field tensor $F_{\mu\nu}(x)$, and indeed when the theory is abelian, it reduces just to the field tensor dotted into the tangent to the loop ξ at the point labelled by the parameter s . Further, it is obvious that $F_\mu[\xi|s]$ is gauge invariant apart from an x -independent rotation at P_0 , which can be easily handled and will henceforth be ignored. And that $F_\mu[\xi|s]$ is sufficient for a full description of the Yang-Mills theory will be clear when one shows that the gauge potential $A_\mu(x)$ is recoverable from $F_\mu[\xi|s]$. One sees thus that $F_\mu[\xi|s]$ does have

⁴Explicitly, for any functional $\Psi[\xi]$ of ξ , the functional derivative is defined as:

$$\frac{\delta}{\delta\xi^\mu(s)} \Psi[\xi] = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \{\Psi[\xi'] - \Psi[\xi]\}, \quad (3.11)$$

with:

$$\xi'^\alpha(s') = \xi^\alpha(s') + \Delta \delta_\mu^\alpha \delta(s - s'), \quad (3.12)$$

where in case of ambiguity, the variation is to be evaluated by first replacing $\delta(s - s')$ with a smooth function of s' peaked at s with width 2η , and then taking the $\eta \rightarrow 0$ limit afterwards.

the credentials we sought for the field variable in the gauge invariant ‘alternative’ formulation of the Yang-Mills theory.

However, there are, of course, vastly more loops in space-time than there are points, so that the variables $\{F_\mu[\xi|s]\}$ which are labelled by loops ξ as well as points s on the loops must be highly redundant when, as one recalls, the theory is already sufficiently described by $\{A_\mu(x)\}$ which are labelled only by the points x in space-time. To use $\{F_\mu[\xi|s]\}$ as variables, they must therefore be severely constrained so as to remove their redundancy, i.e. to ensure that they are all in fact derivable from a potential $A_\mu(x)$ as per (3.10) and (3.9). In other words, we need a parallel to (2.10) of the abelian theory which removed the redundancy in $\{F_{\mu\nu}(x)\}$ and ensured that they can be derived from a potential. This problem has also been solved by a result which we shall refer to here as the *Extended Poincaré Lemma*. [19] This affirms that an $\{A_\mu(x)\}$ can be recovered from a given set $\{F_\mu[\xi|s]\}$ provided that the latter satisfies the following condition:

$$G_{\mu\nu}[\xi|s, s'] = 0, \quad (3.15)$$

where:

$$G_{\mu\nu}[\xi|s, s'] = \delta_\nu(s')F_\mu[\xi|s] - \delta_\mu(s)F_\nu[\xi|s'] + ig[F_\mu[\xi|s], F_\nu[\xi|s']]; \quad (3.16)$$

and provided that $F_\mu[\xi|s]$ is transverse, namely:

$$F_\mu[\xi|s]\dot{\xi}^\mu(s) = 0, \quad (3.17)$$

and does not depend on $\xi(s')$ for $s' > s$. Conversely, it is readily seen that if $F_\mu[\xi|s]$ is derivable from a potential $A_\mu(x)$ in the manner (3.10) and (3.9), then the above three conditions are automatically satisfied.

From (3.10), one sees that the condition (3.17) just says that the longitudinal derivative of $\Phi[\xi]$ with respect to $\xi(s)$ is zero, or that $\Phi[\xi]$ is independent of the parametrisation of ξ . Further, from (3.13), (3.17) is seen to be equivalent just to the statement in the abelian theory that $F_{\mu\nu}(x)$ is antisymmetric under an interchange of its indices. On the other hand, the condition that $F_\mu[\xi|s]$ does not depend on $\xi(s')$ for $s' > s$ is a consequence of the ordering P_s of the exponential in (3.9), and is related to the fact in the abelian theory that $F_{\mu\nu}(x)$ is a local quantity depending only on x . Both these conditions being relatively easy to handle, they will henceforth be regarded as understood and absorbed into the notation for $F_\mu[\xi|s]$.

The other condition (3.15) is more interesting, deserving some scrutiny. Recalling that $F_\mu[\xi|s]$ can be regarded as a ‘connection’ for parallel phase transport in loop space, one sees that $G_{\mu\nu}[\xi|s, s']$ is just the corresponding curvature, which as usual represents the change in phase obtained in parallelly transporting around an infinitesimal closed circuit. In ordinary space, this circuit in loop space may be pictured as a surface where for $s \neq s'$, $G_{\mu\nu}[\xi|s, s']$ is seen to be zero since the surface encloses no volume, but for $s' = s$, the surface becomes a sort of

an infinitesimal version of that Σ which was used to specify a monopole of the abelian theory in the preceding section. Indeed, the same procedure has been used by Lubkin [22], Wu and Yang [18] and Coleman [23] to specify a monopole of Yang-Mills theories in general, as will be discussed in further detail later in Section 5. The condition (3.15) is thus seen to be just the statement that there are no monopoles anywhere in the nonabelian theory, and has thus exactly the same physical content as the Bianchi identity (2.10) of electromagnetism. This has enabled us to apply the Poincaré lemma in the abelian case to remove the redundancy of the variables $F_{\mu\nu}(x)$ and at the same time to recover the potential $A_\mu(x)$, just as (3.15) is meant to do here for the variables $F_\mu[\xi|s]$. One sees therefore that there is indeed a close parallel between the so-called Extended Poincaré Lemma above and the original Poincaré lemma in the abelian theory which justifies the above choice of nomenclature.

With this result, we can now proceed to reformulate Yang-Mills theory in terms of $F_\mu[\xi|s]$. Using (3.13), the action \mathcal{A}_F^0 in (3.2) can be rewritten as:

$$\mathcal{A}_F^0 = -\frac{1}{4\pi\bar{N}} \int \delta\xi ds \operatorname{Tr}\{F_\mu[\xi|s]F^\mu[\xi|s]\}\dot{\xi}(s)^{-2}, \quad (3.18)$$

where the integral is to be taken over all parametrised loops ⁵ ξ and over all points s on each loop, and \bar{N} is an (infinite) normalisation factor defined as:

$$\bar{N} = \int_0^{2\pi} ds \int \prod_{s' \neq s} d^4\xi(s'). \quad (3.19)$$

Equations of motion are to be obtained by extremising (3.18) with respect to $F_\mu[\xi|s]$ under the constraint (3.15). Using as Lagrange multipliers $L_{\mu\nu}[\xi|s]$, we extremise:

$$\mathcal{A}_F = \mathcal{A}_F^0 + \int \delta\xi ds \operatorname{Tr}\{L_{\mu\nu}[\xi|s]G^{\mu\nu}[\xi|s]\}, \quad (3.20)$$

with respect to $F_\mu[\xi|s]$. Notice that $G_{\mu\nu}[\xi|s, s']$ being automatically zero for $s \neq s'$, we need to impose the constraint only for $s = s'$, and the symbol $G_{\mu\nu}[\xi|s]$ in (3.20) is meant to represent $G_{\mu\nu}[\xi|s, s']$ with s' smeared around s .

The extremisation gives:

$$F_\mu[\xi|s] = -(4\pi\bar{N}\dot{\xi}(s)^2)\mathcal{D}^\nu(s)L_{\mu\nu}[\xi|s], \quad (3.21)$$

where $\mathcal{D}_\mu(s)$ is the ‘covariant derivative’ in loop space:

$$\mathcal{D}_\mu(s) = \delta_\mu(s) - ig[F_\mu[\xi|s], \quad]. \quad (3.22)$$

(3.21) is the equation of motion in parametric form, from which the Lagrange multipliers $L_{\mu\nu}[\xi|s]$ can be eliminated [21] to give:

$$\mathcal{D}^\mu(s)F_\mu[\xi|s] = 0. \quad (3.23)$$

⁵We note that parametrised loops ξ being by definition just functions of s , integrals over ξ are just ordinary functional integrals, which is in fact one reason why we prefer to work with parametrised loops rather than the actual loops.

Further, by (3.22), one sees that (3.23) is the same as:

$$\delta^\mu(s)F_\mu[\xi|s] = 0, \quad (3.24)$$

the Polyakov equation, which is known, by (3.13), to be exactly equivalent to the Yang-Mills equation (3.3). The reformulation of the original Yang-Mills problem in terms of the loop variables $F_\mu[\xi|s]$ is now complete.

We next recall that the original reason given above for attempting a reformulation of the theory in terms of $F_\mu[\xi|s]$ was the suspicion that, by analogy with the parallel formulation of the abelian theory as summarised in the second column of Chart I, the Lagrange multiplier $L_{\mu\nu}[\xi|s]$ should play here the role of a dual potential, given a suitable generalisation of the concept of duality. That this is indeed possible can be made more transparent by a few notational rearrangements. We note first that (3.21) can be rewritten as:

$$F_\mu[\xi|s] = -(4\pi\bar{N}\dot{\xi}(s)^2)\Phi^{-1}\delta^\nu(s)\{\Phi L_{\mu\nu}[\xi|s]\Phi^{-1}\}\Phi, \quad (3.25)$$

for any Φ satisfying:

$$\delta_\mu(s)\Phi = -ig\Phi F_\mu[\xi|s], \quad (3.26)$$

which holds for any $\Phi_\xi(s_1, 0)$ defined as in (3.14) with $s_1 > s$. In particular, we may choose Φ to be $\Phi_\xi(s_+, 0)$, where s_+ is $s + \eta$ with $2\eta > 0$ being the ‘width’ of the δ -function in the definition of the functional derivative $\delta^\nu(s)$, which ‘width’ is to be taken to zero afterwards, as explained in the footnote after (3.10). Hence, if one defines:

$$H_{\mu\nu\rho}[\xi|s] = -\frac{1}{\sqrt{6}}\epsilon_{\mu\nu\rho\sigma}\Phi_\xi(s, 0)F^\sigma[\xi|s]\Phi_\xi^{-1}(s, 0), \quad (3.27)$$

as the (generalised) dual to $F_\mu[\xi|s]$, then by (3.25) and (3.26) it is seen to be expressible as:

$$H_{\mu\nu\rho}[\xi|s] = \delta_\mu(s)T_{\nu\rho}[\xi|s] + \delta_\nu(s)T_{\rho\mu}[\xi|s] + \delta_\rho(s)T_{\mu\nu}[\xi|s], \quad (3.28)$$

for:

$$T_{\mu\nu}[\xi|s] = -\sqrt{\frac{2}{3}}\pi\bar{N}\dot{\xi}(s)^2\epsilon_{\mu\nu\rho\sigma}\Phi_\xi(s_+, 0)L^{\rho\sigma}[\xi|s]\Phi_\xi^{-1}(s_+, 0). \quad (3.29)$$

In other word, this dual H to F is here expressed as an exterior derivative of T , which we can thus regard as a generalised dual potential.

For the abelian theory, by (3.13), (3.27) reduces to

$$H_{\mu\nu\rho}[\xi|s] = -\frac{1}{\sqrt{6}}\epsilon_{\mu\nu\rho\sigma}F^{\sigma\alpha}(\xi(s))\dot{\xi}_\alpha(s), \quad (3.30)$$

so that:

$$*F_{\mu\nu}(x) = -\frac{2\sqrt{6}}{\bar{N}}\int\delta\xi ds H_{\mu\nu\rho}[\xi|s]\dot{\xi}^\rho(s)\dot{\xi}(s)^{-2}\delta(x - \xi(s)), \quad (3.31)$$

while (3.28) reduces to (2.6) with:

$$\tilde{A}_\mu(x) = \frac{2\sqrt{6}}{N} \int \delta\xi ds T_{\mu\nu}[\xi|s] \dot{\xi}^\nu(s) \dot{\xi}(s)^{-2} \delta(x - \xi(s)). \quad (3.32)$$

Hence, apart from the fact that in the loop space representation quantities such as $H_{\mu\nu\rho}[\xi|s]$, $T_{\mu\nu}[\xi|s]$ and $F_\mu[\xi|s]$ may carry one more or one less index than their corresponding local quantities ${}^*F_{\mu\nu}(x)$, $A_\mu(x)$ and $F_{\mu\nu}(x)$ due to the extra direction carried by the tangent to the loop, the equations (3.27) and (3.28) are just the generalisations of respectively the abelian duality relationship (2.4) and the definition (2.6) of the dual potential.

Furthermore, the condition (3.28) implies that:

$$\delta^\sigma \epsilon_{\mu\nu\rho\sigma} H^{\mu\nu\rho}[\xi|s] = 0, \quad (3.33)$$

as can be shown by direct calculation, and by (3.27), this is seen to be identical to the Polyakov equation (3.24). Hence, the existence of the dual potential $T_{\mu\nu}[\xi|s]$ guarantees the validity of the equation of motion, which is the parallel to the statement that (2.6) guarantees (2.3) in the abelian theory. Conversely, since the left-hand side of (3.33) can be regarded as the exterior derivative of the 3-form H , its vanishing implies via the Poincaré lemma that there exists a 2-form T of which H is itself the exterior derivative. In other words, the equation of motion (3.24) or (3.33) also implies (3.28), which is the parallel to the statement that (2.3) implies (2.6) in the abelian case.

For electromagnetism, the tensor ${}^*F_{\mu\nu}(x)$ as given in (2.6) was invariant under the transformation (2.8). Similarly here, the tensor $H_{\mu\nu\rho}[\xi|s]$ as given by (3.28) is invariant under the transformation:

$$T_{\mu\nu}[\xi|s] \longrightarrow T_{\mu\nu}[\xi|s] + \delta_{[\nu} \tilde{\Lambda}_{\mu]}. \quad (3.34)$$

Moreover, under the transformation (2.8), the abelian action (2.11) is invariant, as can be seen after an integration by parts from the antisymmetry of ${}^*F_{\mu\nu}(x)$. Under the analogous transformation (3.34) here, the Lagrange multiplier $L_{\mu\nu}[\xi|s]$ transforms as:

$$L_{\mu\nu}[\xi|s] \longrightarrow L_{\mu\nu}[\xi|s] + \sqrt{6}(4\pi\bar{N}\dot{\xi}(s)^2)^{-1} \epsilon_{\mu\nu\rho\sigma} \mathcal{D}^\sigma(s) \{ \Phi_\xi^{-1}(s_+, 0) \tilde{\Lambda}^\rho[\xi|s] \Phi_\xi(s_+, 0) \}. \quad (3.35)$$

We note that, apart from some trivial factors and changes in notation, this is exactly the transformation of the antisymmetric tensor potential suggested by Freedman and Townsend [14] in a different context, and that the action (3.20) has also the form of their invariant action. Indeed, on substituting into (3.20) and integrating by parts with respect to $\delta\xi(s)$, this gives an increment:

$$\Delta\mathcal{A}_F = \sqrt{6}(4\pi\bar{N}\dot{\xi}(s)^2)^{-1} \int \delta\xi ds \text{Tr} \{ \Phi_\xi^{-1}(s_+, 0) \tilde{\Lambda}_\sigma[\xi|s] \Phi_\xi(s_+, 0) \epsilon^{\mu\nu\rho\sigma} \mathcal{D}_\rho(s) G_{\mu\nu}[\xi|s] \}, \quad (3.36)$$

which vanishes because of the Bianchi identity satisfied by $G_{\mu\nu}[\xi|s]$ by virtue of its definition in (3.16). Hence, we conclude that, analogously to the abelian theory, the symmetry in the nonabelian theory is also enlarged by a new symmetry on the dual potential. That this new symmetry should form another $SU(N)$, however, will only be apparent later when considering the interaction of the Yang-Mills field with a colour magnetic charge.

For easier comparison with electromagnetism, the arguments in this section are summarised on the left half of Chart III. As can be seen, the analogy with Chart I for the abelian theory is rather close.

4 Pure Yang-Mills Theory - Dual Formulation

In electromagnetism, dual symmetry means that electricity and magnetism can be inverted so that one can reformulate the theory entirely in terms of $*F_{\mu\nu}(x)$ and $\tilde{A}_\mu(x)$ as variables rather than $F_{\mu\nu}(x)$ and $A_\mu(x)$. Hence, in the nonabelian theory, having introduced now a generalised dual relationship with $H_{\mu\nu\rho}[\xi|s]$ and $T_{\mu\nu}[\xi|s]$ as respectively the ‘dual field’ and ‘dual potential’, one would hope to be able also to reformulate the theory in a dual fashion, starting with $H_{\mu\nu\rho}[\xi|s]$ and $T_{\mu\nu}[\xi|s]$ as variables instead of $F_\mu[\xi|s]$ and $A_\mu(x)$. Further, in analogy to the direct formulation, one would expect that the formulation can be carried out in two ways: either by expressing the action in terms of the potential $T_{\mu\nu}[\xi|s]$ and extremising with respect to $T_{\mu\nu}[\xi|s]$, or else by adopting the \tilde{U} -invariants $H_{\mu\nu\rho}[\xi|s]$ as variables but imposing appropriate constraints on them so as to ensure that they can be derived from a potential. In either case, one would expect to obtain exactly the same result as in the direct formulation of the theory.

In terms of the variables $H_{\mu\nu\rho}[\xi|s]$, the field action (3.2) or (3.18) can be rewritten as:

$$\mathcal{A}_F^0 = \frac{1}{4\pi N} \int \delta\xi ds \text{Tr}\{H_{\mu\nu\rho}[\xi|s]H^{\mu\nu\rho}[\xi|s]\}\dot{\xi}(s)^{-2}. \quad (4.1)$$

Adopting the first approach, we substitute the expression (3.28) into (4.1) and extremise with respect to $T_{\mu\nu}[\xi|s]$. This gives:

$$\delta_\rho(s)H^{\mu\nu\rho}[\xi|s] = 0, \quad (4.2)$$

as the equation of motion. In analogy to the direct formulation of pure Yang-Mills theory as summarised in the first column of Chart III, as well as to the dual formulation of the abelian theory as summarised in the last column of Chart I, one would expect this equation (4.2) to guarantee the existence of the (direct) potential $A_\mu(x)$. That this is the case can be seen as follows. The loop space curvature $G_{\mu\nu}[\xi|s, s']$ in (3.16) can be expressed as:

$$\begin{aligned} G_{\mu\nu}[\xi|s, s'] &= \Phi_\xi^{-1}(s, 0)\delta_\nu(s')\{\Phi_\xi(s, 0)F_\mu[\xi|s]\Phi_\xi^{-1}(s, 0)\}\Phi_\xi(s, 0) \\ &\quad - \Phi_\xi^{-1}(s', 0)\delta_\mu(s)\{\Phi_\xi(s', 0)F_\nu[\xi|s']\Phi_\xi^{-1}(s', 0)\}\Phi_\xi(s', 0), \end{aligned} \quad (4.3)$$

since, by (3.26):

$$\begin{aligned} & \delta_\nu(s') \{ \Phi_\xi(s, 0) F_\mu[\xi|s] \Phi_\xi^{-1}(s, 0) \} \\ &= \Phi_\xi(s, 0) \{ \delta_\nu(s') F_\mu[\xi|s] - ig[F_\nu[\xi|s'], F_\mu[\xi|s]] \theta(s - s') \} \Phi_\xi^{-1}(s, 0), \end{aligned} \quad (4.4)$$

and a similar relation with $s \leftrightarrow s', \mu \leftrightarrow \nu$. Hence, by (3.27), we have, on putting $s = s'$:

$$G_{\mu\nu}[\xi|s] = -\sqrt{\frac{3}{2}} \epsilon_{\mu\nu\rho\sigma} \Phi_\xi^{-1}(s, 0) \delta_\alpha(s) H^{\rho\sigma\alpha}[\xi|s] \Phi_\xi(s, 0) = 0, \quad (4.5)$$

which implies by the Extended Poincaré Lemma in (3.15) that $A_\mu(x)$ exists and is related to $H_{\mu\nu\rho}[\xi|s]$ by (3.27), (3.13) and (3.14) as required. We have thus completed the analogy here to the direct formulation summarised in the first column of Chart III.

Suppose we wish now to adopt instead the \tilde{U} -invariants $H_{\mu\nu\rho}[\xi|s]$ as variables, which, as we have seen in the abelian theory, are more convenient when we later deal with monopoles. In that case, we need to impose on $H_{\mu\nu\rho}[\xi|s]$ appropriate constraints so as to ensure that $H_{\mu\nu\rho}[\xi|s]$ is indeed derivable from a potential $T_{\mu\nu}[\xi|s]$ as it should. The appropriate constraints are just those exhibited in (3.33) because of the Poincaré lemma in loop space, as was explained there. This is the dual image of the constraint $G_{\mu\nu}[\xi|s] = 0$ in the direct formulation (Chart III, second column) and the parallel of the constraint $\partial_\nu F^{\mu\nu}(x) = 0$ in the dual formulation of the abelian theory (Chart I, third column).

Hence, we write:

$$\tilde{\mathcal{A}}_F = \tilde{\mathcal{A}}_F^0 + \int \delta\xi ds \operatorname{Tr} \{ L[\xi|s] \delta_\mu \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}[\xi|s] \}, \quad (4.6)$$

with $\tilde{\mathcal{A}}_F^0$ given as in (4.1), and $L[\xi|s]$ as the Lagrange multiplier. This is to be extremised with respect to $H_{\mu\nu\rho}[\xi|s]$, which gives as the equation of motion:

$$H_{\mu\nu\rho}[\xi|s] = -(2\pi \bar{N} \dot{\xi}(s)^2) \epsilon_{\mu\nu\rho\sigma} \delta^\sigma L[\xi|s], \quad (4.7)$$

in terms of the Lagrange multiplier $L[\xi|s]$ as parameter. This is the dual image of (3.21) in the direct formulation.

Extending the analogy further, one would expect that (4.7) would imply the equation of motion (4.2) derived before, and that the Lagrange multiplier $L[\xi|s]$ would play the role of the ‘dual potential’ for $H_{\mu\nu\rho}[\xi|s]$, or in other words the potential for $F_\mu[\xi|s]$, namely $A_\mu(x)$. That (4.7) implies (4.2) is straightforward, since:

$$\delta_\rho H^{\mu\nu\rho}[\xi|s] = -(2\pi \bar{N} \dot{\xi}(s)^2) \epsilon^{\mu\nu\rho\sigma} \delta_\rho(s) \delta_\sigma(s) L[\xi|s], \quad (4.8)$$

vanishes by symmetry, and as already noted, this implies via the vanishing of $G_{\mu\nu}[\xi|s]$ that a potential $A_\mu(x)$ exists. That this $A_\mu(x)$ should be given in terms of the Lagrange multiplier $L[\xi|s]$ is a little more involved but can be seen as follows.

Figure 1: Illustration for the recovery of $A_\mu(x)$ from $W[\xi|s]$

Let us first, for convenience, normalise $L[\xi|s]$ by a factor, calling it:

$$W[\xi|s] = \sqrt{6}(2\pi\bar{N}\dot{\xi}(s)^2)L[\xi|s], \quad (4.9)$$

so that by (4.8) above:

$$F_\mu[\xi|s] = \Phi_\xi^{-1}(s, 0)\delta_\mu(s)W[\xi|s]\Phi_\xi(s, 0). \quad (4.10)$$

Define then:

$$A_\mu(x) = \frac{2}{\bar{N}} \int \delta\xi ds W[\xi|s] \dot{\xi}_\mu(s) \dot{\xi}(s)^{-2} \delta(x - \xi(s)), \quad (4.11)$$

which we wish to show is the actual potential we want. With this $A_\mu(x)$ in (4.11) as connection, one can define as usual by (3.9) a phase factor $\Phi[\xi]$ for any parametrised closed loop ξ beginning and ending at the reference point P_0 . For every point x in space-time, choose a path γ_x linking x to P_0 in such a way that neighbouring points will be linked to P_0 by neighbouring paths. Thus, in particular, γ_x may be chosen just as the straight line joining x to P_0 . Given any loop ξ then, the two points on it, $\xi(s_+)$ and $\xi(s_-)$ for $s_\pm = s \pm \eta$, specify a wedge-shaped closed loop $\gamma_{\xi(s_+)}^{-1} \circ \xi(s_+, s_-) \circ \gamma_{\xi(s_-)}$, as depicted in Figure 1, where $\xi(s_+, s_-)$ denotes the segment of ξ between s_- and s_+ . For this closed loop, one can construct the standard phase factor (3.9), an element of the gauge group which we denote as:

$$g(\xi(s_+), \xi(s_-)) = \Phi[\gamma_{\xi(s_+)}^{-1} \circ \xi(s_+, s_-) \circ \gamma_{\xi(s_-)}]. \quad (4.12)$$

Then, by the standard loop space procedure for constructing local potentials, we conclude that:

$$A_\mu(\xi(s))\dot{\xi}^\mu(s) \sim -\frac{i}{g} \lim_{\eta \rightarrow 0} \frac{1}{2\eta} \{g(\xi(s_+), \xi(s_-)) - 1\} \quad (4.13)$$

Figure 2: Differentiating the wedge

where \sim means that the two sides are equal up to an ordinary Yang-Mills gauge transformation. [An outline of a proof for (4.13) can be found, for example, in ref. [19]] Further, as is well known, a gauge transformation here is the same as a choice of the path γ_x linking each point x in space-time to the reference point P_0 . Hence by the appropriate choice of γ_x , which we know exist but need not specify, we can actually replace the sign \sim above by an equality and write instead:

$$A_\mu(\xi(s))\dot{\xi}^\mu(s) = -\frac{i}{g} \lim_{\eta \rightarrow 0} \frac{1}{2\eta} \{g(\xi(s_+), \xi(s_-)) - 1\}, \quad (4.14)$$

where, we recall, $g(\xi(s_+), \xi(s_-))$ is the phase factor Φ in (3.9) for the wedge-shaped loop $\gamma_{\xi(s_+)}^{-1} \circ \xi(s_+, s_-) \circ \gamma_{\xi(s_-)}$. If we put then:

$$W[\xi|s] = -\frac{i}{g} \frac{1}{\eta} \{g(\xi(s_+), \xi(s_-)) - 1\}, \quad (4.15)$$

we see we recover $A_\mu(x)$ as defined in (4.11).

We have yet to show that the $W[\xi|s]$ so constructed does indeed satisfy the equation (4.10). This is best seen diagrammatically. Differentiating the wedge in Figure 1 with respect to $\xi^\mu(s)$, one obtains Figure 2. Multiplying next by the factors $\Phi_\xi^{-1}(s_+, 0)$ and $\Phi_\xi(s_+, 0)$ gives Figure 3, which is exactly $F_\mu[\xi|s]$ as defined in (3.10). This completes then our demonstration that the quantity defined in (4.11) in terms of $W[\xi|s]$ or $L[\xi|s]$ is indeed the potential $A_\mu(x)$ we sought.

We have shown now not only that the action when constrained by (3.33) leads to the correct equation of motion but also that the Lagrange multiplier $L[\xi|s]$ gives via (4.11) the local potential, in parallel to (2.30) for the abelian theory. It remains to show only that the action $\tilde{\mathcal{A}}_F$ is invariant under a gauge transformation

Figure 3: Obtaining $F_\mu[\xi|s]$ by differentiating the wedge

of $A_\mu(x)$. As usual, of course, $A_\mu(x)$ transforms under an infinitesimal gauge transformation as:

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \Lambda(x) + ig[\Lambda(x), A_\mu(x)]. \quad (4.16)$$

Under such a transformation, $F_\mu[\xi|s]$ is invariant, and $\Phi_\xi(s, 0)$ transforms as:

$$\Phi_\xi(s, 0) \longrightarrow [1 + ig\Lambda(\xi(s))]\Phi_\xi(s, 0) \quad (4.17)$$

so that $H_{\mu\nu\rho}[\xi|s]$ is covariant, which implies that $\tilde{\mathcal{A}}_F^0$ in (4.1) is invariant under this U -transformation. For $\tilde{\mathcal{A}}_F$ in (4.6) to be invariant, we want therefore that the second term in (4.6) due to the constraint be also invariant. From (4.15), and recalling the general transformation properties of phase factors, one sees that $W[\xi|s]$ transforms as:

$$W[\xi|s] \longrightarrow \{1 + ig\Lambda(\xi(s_+))\}W[\xi|s]\{1 - ig\Lambda(\xi(s_-))\} + \frac{1}{\eta}\{\Lambda(\xi(s_+)) - \Lambda(\xi(s_-))\}. \quad (4.18)$$

Substituting this into the second term of $\tilde{\mathcal{A}}_F$ in (4.6) and performing an integration by parts with respect to ξ , one sees that the derivative $\delta_\sigma(s)$ does not operate on the local quantities $\Lambda(\xi(s_\pm))$ defined at the end-points of the interval (s_+, s_-) so that the second term in (4.18) gives zero variation. One obtains then just:

$$\begin{aligned} - \int \delta\xi ds \quad & \text{Tr} \quad \{[1 + ig\Lambda(\xi(s_+))]\delta_\sigma(s)L[\xi|s][1 - ig\Lambda(\xi(s))]\} \\ & \times [1 + ig\Lambda(\xi(s))]\epsilon_{\mu\nu\rho\sigma}H^{\mu\nu\rho}[\xi|s][1 - ig\Lambda(\xi(s))]\} \end{aligned} \quad (4.19)$$

which gives the same value as before when $s_+ \rightarrow s_-$. The action $\tilde{\mathcal{A}}_F$ is thus indeed invariant as required.

The result of this section is summarised on the right-hand side of Chart III, which is seen to be quite a close dual image of the direct formulation on the left, and closely parallels also the dual formulation of the abelian theory on the right of Chart I.

5 Yang-Mills Field with Colour Magnetic Charge

By a *colour magnetic charge* we mean here just a monopole of the Yang-Mills field in the same way that a magnetic charge in electromagnetism is understood as a monopole of the Maxwell field, but since the term ‘monopole’ has been used to mean quite different things in a wide variety of contexts, we need to be more specific. Let us first recall the Lubkin [22], Wu-Yang [18], Coleman [23] definition of a monopole as a topological obstruction in a Yang-Mills field. As for the abelian theory, start again with a family of closed loops enveloping a surface Σ as that considered in Section 2, and for each loop C_t , suitably parametrised by a function, say $\xi_t^\mu(s) : s = 0 \rightarrow 2\pi$, construct the phase factor $\Phi[\xi_t]$ as in (3.9). For each t , this gives an element of the gauge group G , so that as t varies from 0 to 2π , $\Phi[\xi_t]$ traces, as in the abelian case, a closed curve Γ_Σ in the gauge group beginning and ending at the identity element. Depending on the topology of the gauge group, it may then again happen, as in the abelian theory, that Γ_Σ cannot be continuously deformed to a point even when Σ itself shrinks to zero. We have then a topological obstruction inside Σ , which, by analogy to the abelian theory, is designated a monopole. In mathematical terms, the monopole charge is defined as the homotopy class of Γ_Σ in G , which is thus automatically quantised and conserved.

In electromagnetism, a magnetic charge may also be regarded as a source of the dual Maxwell field $*F_{\mu\nu}(x)$ so that its dynamics can be formulated in terms of the dual field in exactly the same way that the dynamics of a electric charge, when considered as a source of $F_{\mu\nu}(x)$, is formulated in terms of the Maxwell field. For Yang-Mills theory, however, for lack of a known dual symmetry, this need no longer be the case. A colour magnetic charge as defined above is not a source of the dual Yang-Mills field $*F_{\mu\nu}(x)$ so that its dynamics cannot be formulated in the same way as the dynamics of the colour (electric) charge is usually formulated, in which the latter is taken to be a source of the Yang-Mills field.

On the other hand, intuitive arguments were given above in the abelian theory for an intrinsic coupling between a field and its monopole by virtue of the latter’s definition as a topological obstruction of the former. The same intuitive argument is applicable also to a monopole in a nonabelian Yang-Mills field and should lead to a formulation of its dynamics. Hence, in parallel to the development of the abelian theory as summarised on the left of Chart II, we should be able to derive via the same Wu-Yang criterion the equations of motion for a colour magnetic charge from its definition as a monopole of the Yang-Mills field. For that purpose, we ought

again to start with the action of the free field plus the action of a free particle, and impose the condition that the particle should carry a monopole charge of the Yang-Mills field. Extremising the free action then under the defining constraint of the monopole should automatically lead to coupled equations representing the interactions between the colour magnetic charge with the Yang-Mills field. The solution to this problem has already been reported elsewhere.[21] We need here only to summarise the main arguments and results so as to make clear their relationship with the rest of the paper.

As in the abelian theory, the variational problem posed above is hard to solve with the gauge potential as field variables since in the presence of a monopole the gauge potential has to be patched, with the patching dependent on the position of the monopole. For the abelian problem, this difficulty was by-passed by adopting instead the gauge invariant $F_{\mu\nu}(x)$ as field variables. Besides avoiding the complexities of patching, $F_{\mu\nu}(x)$ has the added virtue of giving a simple expression for the monopole charge, and hence a convenient form for the topological constraint we wish to impose.

For the nonabelian theory the field tensor $F_{\mu\nu}(x)$ itself will not do, but it turns out that the Polyakov loop variable $F_\mu[\xi|s]$ introduced in Section 3 is eminently suitable for this problem since, in contrast to $F_{\mu\nu}(x)$, it is patch-independent even in the presence of a monopole and gives an explicit expression for the monopole charge which was defined above only abstractly as a homotopy class.

To see this, we recall from Section 3 that $F_\mu[\xi|s]$ may be regarded as a connection in loop space for which one can construct the curvature $G_{\mu\nu}[\xi|s, s']$ in (3.16). It was further indicated there that $G_{\mu\nu}[\xi|s]$ represents the phase change under parallel transport over an infinitesimal rectangle in loop space, which in ordinary space appears as an infinitesimal version of the surface Σ used in Section 2 to define a monopole. It is thus clear that $G_{\mu\nu}[\xi|s]$ measures in some way the monopole charge at the point $\xi(s)$ enclosed by this infinitesimal surface. The question in exactly what way it measures the charge is perhaps not entirely obvious, but has already been analysed in detail elsewhere. [19] For our purpose here, we need only quote the result for the $SO(3)$ theory where the monopole charge may be denoted by a sign \pm , (i.e. $-$ represents a monopole but $+$ no monopole). Then if the surface representing $G_{\mu\nu}[\xi|s]$ encloses a monopole, we have $G_{\mu\nu}[\xi|s] = \kappa$ at $\xi(s)$ but zero elsewhere, where κ satisfies:

$$\exp i\pi\kappa = -1. \quad (5.1)$$

Suppose now we have a classical point particle with mass m moving along a world-path $Y^\mu(\tau)$. The defining constraint specifying that $Y^\mu(\tau)$ should carry a colour magnetic charge can then be written in a convenient form as:

$$G_{\mu\nu}[\xi|s] = -4\pi J_{\mu\nu}[\xi|s], \quad (5.2)$$

where:

$$J_{\mu\nu}[\xi|s] = \tilde{g} \int d\tau \kappa[\xi|s] \epsilon_{\mu\nu\rho\sigma} \frac{dY^\rho(\tau)}{d\tau} \dot{\xi}^\sigma(s) \delta(\xi(s) - Y(\tau)) \quad (5.3)$$

may be regarded as the monopole current, only carrying an extra index here because in loop space notation, the tangent $\dot{\xi}(s)$ to the loop at s defines an additional direction. (5.2) is the direct nonabelian analogue of the Gauss Law constraint (2.22) in the abelian case. According then to the Wu-Yang criterion, the equations of motion of the colour magnetic charge are to be obtained by extremising the free action under this defining constraint.

The free action for the field-particle system is:

$$\mathcal{A}^0 = \mathcal{A}_F^0 - m \int d\tau, \quad (5.4)$$

where, in terms of $F_\mu[\xi|s]$ as variables, \mathcal{A}_F^0 takes now the form (3.18). Again, as in the loop formulation of the pure Yang-Mills theory in Section 3, $F_\mu[\xi|s]$, being a redundant set as field variables, have to be constrained so as to ensure that the potential $A_\mu(x)$ can be recovered from them. But, as in the parallel problem for the magnetic charge in the abelian theory, the beauty of the present formulation is that the defining constraint (5.2) implies that $G_{\mu\nu}[\xi|s]$ vanishes whenever $\xi(s)$ does not lie on the monopole world-line $Y^\mu(\tau)$ and hence already ensures by the Extended Poincaré Lemma that $A_\mu(x)$ can be recovered everywhere except on $Y^\mu(\tau)$ where the potential is in any case not expected to exist. Hence, so long as (5.2) is imposed we are allowed to adopt $F_\mu[\xi|s]$ as variables and just extremise:

$$\mathcal{A} = \mathcal{A}^0 + \int \delta\xi ds \text{Tr}[L_{\mu\nu}[\xi|s]\{G^{\mu\nu}[\xi|s] + 4\pi J^{\mu\nu}[\xi|s]\}] \quad (5.5)$$

with respect to $F_\mu[\xi|s]$ and $Y^\mu(\tau)$, where $L_{\mu\nu}[\xi|s]$ are the Lagrange multipliers for the defining constraint (5.2).

Extremising \mathcal{A} in (5.5) with respect to $F_\mu[\xi|s]$, one obtains (3.21) as in the pure field theory leading again to the Polyakov form (3.24) of the Yang-Mills equation. Extremising \mathcal{A} with respect to $Y^\mu(\tau)$, one has:

$$\begin{aligned} m \frac{d^2 Y^\mu(\tau)}{d\tau^2} &= -8\pi\tilde{g} \int \delta\xi ds \epsilon^{\mu\nu\rho\sigma} \delta^\lambda(s) \text{Tr}\{L_{\lambda\rho}[\xi|s]\kappa[\xi|s]\} \\ &\quad \frac{dY_\nu(\tau)}{d\tau} \dot{\xi}_\sigma(s) \delta(\xi(s) - Y(\tau)), \end{aligned} \quad (5.6)$$

which, together with (3.24) and the constraint (5.2), constitute the equations of motion for the colour magnetic charge interacting with the Yang-Mills field.

The equation (5.6) may be written in a more familiar form by eliminating the Lagrange multiplier $L_{\mu\nu}[\xi|s]$ using the constraint (5.2), giving:

$$m \frac{d^2 Y^\mu(\tau)}{d\tau^2} = -\frac{2\tilde{g}}{N} \int \delta\xi ds \epsilon^{\mu\nu\rho\sigma} \text{Tr}\{\kappa[\xi|s]F_\nu[\xi|s]\} \frac{dY_\rho(\tau)}{d\tau} \dot{\xi}_\sigma(s) \dot{\xi}(s)^{-2} \delta(\xi(s) - Y(\tau)), \quad (5.7)$$

where on substituting the local expression (3.13) for $F_\mu[\xi|s]$, one has:

$$m \frac{d^2 Y^\mu(\tau)}{d\tau^2} = -\tilde{g} \text{Tr}\{K(\tau)^* F^{\mu\nu}(Y(\tau))\} \frac{dY_\nu(\tau)}{d\tau}, \quad (5.8)$$

with:

$$K(\tau) = \Phi_\xi(s, 0) \kappa[\xi|s] \Phi_\xi^{-1}(s, 0)|_{\xi(s)=Y(\tau)}, \quad (5.9)$$

a local quantity depending only on $Y(\tau)$. With this, the analogy between the first column of Charts II and IV is complete.

One notices that (5.8) is very similar in form to the dual Lorentz equation (2.25) for the abelian magnetic charge. It is also exactly the dual of the equation deduced by Wong [15] by taking the classical limit of the Yang-Mills equation and can be obtained from the Wong equation simply by changing $g \rightarrow \tilde{g}$, $I(\tau) \rightarrow K(\tau)$, and $F_{\mu\nu}(x) \rightarrow {}^*F_{\mu\nu}(x)$. This does not mean, however, that the dynamics of a (nonabelian) colour magnetic charge need be exactly the same as that of a colour (electric) charge as was true for the abelian case. The reason is that if so the constraint (5.2) would have to be the dual of the second Wong equation:

$$D_\nu F^{\mu\nu}(x) = -4\pi g \int d\tau I(\tau) \frac{dY^\mu(\tau)}{d\tau} \delta(x - Y(\tau)) \quad (5.10)$$

but it is not known to be so.

Next, consider a Dirac particle carrying a colour magnetic charge in parallel to the abelian case as summarised in the second column of Chart II. As for the abelian theory we just replace the action (5.4) by:

$$\mathcal{A}^0 = \mathcal{A}_F^0 + \int d^4x \bar{\psi}(x) (i\partial_\mu \gamma^\mu - m) \psi(x), \quad (5.11)$$

and the classical current (5.3) by its Dirac analogue:

$$J_{\mu\nu}[\xi|s] = \tilde{g} \epsilon_{\mu\nu\rho\sigma} [\bar{\psi}(\xi(s)) \gamma^\rho t^i \psi(\xi(s))] \Omega_\xi^{-1}(s, 0) t_i \Omega_\xi(s, 0) \dot{\xi}^\sigma(s), \quad (5.12)$$

where:

$$\Omega_\xi(s, 0) = \omega(\xi(s_+)) \Phi_\xi(s_+, 0) \quad (5.13)$$

is a rotation matrix transforming from the colour electric U -frame at the reference point P_0 in which $J_{\mu\nu}[\xi|s]$ is measured to the colour magnetic \tilde{U} -frame at $\xi(s)$ in which $\psi(\xi(s))$ is measured. The factor $\Phi_\xi(s_+, 0)$ transports the U -frame at P_0 along the loop ξ to the local U -frame at $\xi(s)$, while $\omega(\xi(s))$, which is a local quantity, rotates the U -frame at $x = \xi(s)$ to the \tilde{U} -frame at the same point. The fact that the values of these quantities should be taken at s_+ will only be of relevance later.

Equations of motion can now be derived by extremising the action (5.5) with respect to $F_\mu[\xi|s]$ and $\psi(x)$. The equation obtained from varying $F_\mu[\xi|s]$ is the same as before, while that from varying $\psi(x)$ can be written as:

$$(i\partial_\mu \gamma^\mu - m) \psi(x) = -\tilde{g} \tilde{A}_\mu(x) \gamma^\mu \psi(x), \quad (5.14)$$

where:

$$\tilde{A}_\mu(x) = -\frac{2\sqrt{6}}{N} \int \delta\xi ds \omega(\xi(s_+)) T_{\mu\nu}[\xi|s] \omega^{-1}(\xi(s_+)) \dot{\xi}^\nu(s) \dot{\xi}^\sigma(s)^{-2} \delta(x - \xi(s)). \quad (5.15)$$

The result is again seen to be exactly dual to the corresponding Yang-Mills equation for colour (electric) charges if one regards $\tilde{A}_\mu(x)$ as the ‘dual’ of $A_\mu(x)$, but, as noted already for the classical case above, this need not mean that the dynamics of a colour magnetic and colour electric charges will be the same since $\tilde{A}_\mu(x)$ is not related to ${}^*F_{\mu\nu}(x)$ in the same manner as $A_\mu(x)$ is related to $F_{\mu\nu}(x)$.

In parallel with the last entry in the second column of Chart II for the abelian theory, we expect that the present system should also have a \tilde{U} -invariance. That this is indeed the case is seen if one couples the transformation (3.34) for $T_{\mu\nu}[\xi|s]$ with the following transformation for $\tilde{\psi}(x)$:

$$\psi(x) \longrightarrow [1 + i\tilde{g}\tilde{\Lambda}(x)]\psi(x) \quad (5.16)$$

and for the rotation matrix $\omega(x)$:

$$\omega(x) \longrightarrow [1 + i\tilde{g}\tilde{\Lambda}(x)]\omega(x) \quad (5.17)$$

with:

$$\tilde{\Lambda}(x) = \frac{4\sqrt{6}}{N} \int \delta\xi ds \omega(\xi(s_+)) \tilde{\Lambda}_\nu[\xi|s] \omega^{-1}(\xi(s_+)) \dot{\xi}^\nu(s) \dot{\xi}(s)^{-2} \delta(x - \xi(s)). \quad (5.18)$$

Now under (3.34) for $T_{\mu\nu}[\xi|s]$, the pure field terms in the action \mathcal{A} of (5.5) remain invariant, while under the simultaneous action of (3.34), (5.16) and (5.17), the increment from the second term in (5.11) will cancel the increment from the $\text{Tr}\{L_{\mu\nu}J^{\mu\nu}\}$ term in (3.20) yielding thus overall invariance.

The \tilde{U} -transformation appears abelian on the dual potential $T_{\mu\nu}[\xi|s]$, but its action on $\psi(x)$, $\omega(x)$ and $A_\mu(x)$ is nonabelian. Indeed, by (5.15), one sees that $\tilde{A}_\mu(x)$ transforms as:

$$\tilde{A}_\mu(x) \longrightarrow \tilde{A}_\mu(x) + \partial_\mu \tilde{\Lambda}(x) + i\tilde{g}[\tilde{\Lambda}(x), \tilde{A}_\mu(x)], \quad (5.19)$$

i.e. exactly as a connection should. Although we have given above explicitly only the infinitesimal form of the \tilde{U} -transformations on $\psi(x)$, $\omega(x)$ and $\tilde{A}_\mu(x)$, it is clear that by repeating these transformations (5.16), (5.17) and (5.19), a $SU(N)$ symmetry will be generated. The full symmetry of the field-particle system is thus $SU(N) \times \widetilde{SU(N)}$ as anticipated, but again, as in the abelian case, since $A_\mu(x)$ and $\tilde{A}_\mu(x)$ are related, this does not mean that the symmetries represent two independent degrees of freedom. The transformation on $T_{\mu\nu}[\xi|s]$, however, remains abelian in appearance, with $\widetilde{T_{\mu\nu}[\xi|s]}$ forming thus an (unfaithful) abelian representation of the nonabelian $SU(N)$ group.

With this observation, the parallel with the second column of Chart II for the abelian theory is now also complete.

An interesting fact to note is that although $\tilde{A}_\mu(x)$ plays here the role of parallel phase transport for the wave functions of colour magnetic charges just as the ordinary Yang-Mills potential $A_\mu(x)$ is the parallel phase transport for wave

functions of colour electric charges, $\tilde{A}_\mu(x)$ is not known as yet to share the other function of $A_\mu(x)$ as being the potential from which all other field quantities can be derived. Rather, it seems at present that it is the 2-indexed loop quantity $T_{\mu\nu}[\xi|s]$ which has usurped that function in the dual formulation. We shall return to this interesting point briefly in Section 7.

6 Yang-Mills Field with Colour (Electric) Charge

In standard Yang-Mills theory, a Dirac particle carrying a colour (electric) charge is introduced as a source of the nonabelian gauge field, the dynamics of which is usually formulated as follows. We add to the free action (5.11) an interaction term suggested by the minimal coupling hypothesis:

$$\mathcal{A}_I = g \int d^4x \bar{\psi}(x) A_\mu(x) \gamma^\mu \psi(x). \quad (6.1)$$

Then, extremising the total action $\mathcal{A}_0 + \mathcal{A}_I$ with respect to the variables $A_\mu(x)$ and $\psi(x)$, we obtain as equations of motion the standard Yang-Mills equations:

$$D_\nu F^{\mu\nu}(x) = -4\pi g \bar{\psi}(x) \gamma^\mu \psi(x), \quad (6.2)$$

and:

$$(i\partial_\mu \gamma^\mu - m)\psi(x) = -g A_\mu(x) \gamma^\mu \psi(x), \quad (6.3)$$

where, of course, $F_{\mu\nu}(x)$ is understood to be derivable from $A_\mu(x)$ as per (3.1). In the case of a classical point particle, we have instead as equations of motion the Wong equations, namely (5.10) and:

$$m \frac{d^2 Y^\mu(\tau)}{d\tau^2} = -g \text{Tr}\{I(\tau) F^{\mu\nu}(Y(\tau))\} \frac{dY_\nu(\tau)}{d\tau}, \quad (6.4)$$

which were obtained as the classical limit of the corresponding Yang-Mills equations, namely (6.2) and (6.3) respectively. Surprisingly, however, the action corresponding to these Wong equations is unknown, and it appears that even the concept of an action may have to be generalised before the Wong equations can be so accommodated in the conventional approach to the dynamics of a colour (electric) charge. [24]

On the other hand, analogy to electromagnetic duality as summarised in Chart II suggests that there may be an alternative approach to formulating the dynamics of a colour electric charge moving in a nonabelian gauge field, namely to consider, if possible, the charge instead as a monopole of the dual field. In that case, the Wu-Yang criterion can lead to the equations of motion as a unique consequence by extremising the free action under the defining constraint of the monopole. And, if the previous examples are any guide, this should work both for the classical point particle and for the Dirac particle. Our purpose here is to study whether such a procedure actually applies and whether it leads to the correct equations.

Our first question then is whether a colour electric charge in Yang-Mills theory can indeed be considered as a monopole of the ‘dual field’ $H_{\mu\nu\rho}[\xi|s]$. It is, however, not entirely clear what a monopole of $H_{\mu\nu\rho}[\xi|s]$ means since $H_{\mu\nu\rho}[\xi|s]$ is a quantity the likes of which we have not met with before, so that we can proceed only by analogy to previous examples. Now in all cases so far considered, a monopole emerged as a topological obstruction in a construct as that detailed in Section 2, where one had first to define a holonomy with respect to some connection giving the parallel transport for the phases of wave functions. In the preceding section, for example, when considering a colour magnetic charge as a monopole of the Yang-Mills field, the connection employed was the Yang-Mills potential $A_\mu(x)$ which gave the parallel transport for the phases of the wave functions $\psi(x)$ of colour electric charges. So by analogy, now that we are attempting to treat colour electric charges as monopoles, we would want a connection which gives the parallel transport for the phases of the wave functions of colour magnetic charges. But in the preceding section, we have already met with a quantity which seems to fulfill this role, namely $\tilde{A}_\mu(x)$ in (5.15), which transformed under \tilde{U} -transformations in (5.19) exactly as a \tilde{U} -connection should. Suppose that we were to take this $\tilde{A}_\mu(x)$ as connection and construct with it a phase factor (holonomy) in exactly the same manner as (3.9), then following the procedure of Lubkin [22], Wu and Yang [18], and Coleman [23] as detailed in Section 2, we should find again topological obstructions which ought by rights to be regarded as the monopoles dual to the colour magnetic charges. And if our conjecture above were indeed correct, then these monopoles in $\tilde{A}_\mu(x)$ should represent none other than colour electric charges.

On the other hand, colour electric charges are already defined through the current they carry in, for example, the Wong equation (5.10). In terms of the variables $H_{\mu\nu\rho}[\xi|s]$, this equation can be rewritten via (3.13) and (3.27) as:

$$\delta_\mu \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}[\xi|s] = -4\sqrt{6}\pi g \int d\tau I(\tau) \frac{dY^\mu(\tau)}{d\tau} \dot{\xi}_\mu(s) \delta(\xi(s) - Y(\tau)). \quad (6.5)$$

Thus the question of whether a colour electric charge can be considered as a monopole in $\tilde{A}_\mu(x)$ becomes whether the equation (6.5) is equivalent to the statement that there is a monopole of $\tilde{A}_\mu(x)$ on the world-line $Y^\mu(\tau)$ in (6.5).

That this is indeed the case can be seen as follows, using arguments closely echoing those that one would use in the abelian theory to show that an electric charge is a monopole of $*F_{\mu\nu}(x)$ by virtue of the Maxwell equation (2.26). First, multiplying (6.5) by $\dot{\xi}^\alpha(s) \xi(s)^{-2} \delta(x - \xi(s))$ and integrating over $\delta\xi ds$, one obtains:

$$\begin{aligned} & \int \delta\xi \quad ds \quad \delta_\mu(s) \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}[\xi|s] \dot{\xi}^\alpha(s) \xi(s)^{-2} \delta(x - \xi(s)) \\ &= -\sqrt{6}\pi \bar{N} g \int d\tau I(\tau) \frac{dY^\alpha(\tau)}{d\tau} \delta(x - Y(\tau)). \end{aligned} \quad (6.6)$$

This equation being Lorentz invariant, we can choose, without loss of generality, the direction of $Y^\alpha(\tau)$ at τ to be along the 0th direction. Next, from the fact that

space-time is 4-dimensional, and that the loop quantity $H_{\mu\nu\rho}[\xi|s]$ has by definition non-vanishing derivatives only in directions transverse to the loop, namely:

$$\delta_\alpha(s)H_{\mu\nu\rho}[\xi|s]\dot{\xi}^\alpha(s) = 0, \quad (6.7)$$

one can show that:

$$\delta^\sigma(s)\epsilon_{\mu\nu\rho\sigma}H^{\mu\nu\rho}[\xi|s]\dot{\xi}_0(s) = 3\delta^i(s)\epsilon_{ijk}H^{\mu jk}[\xi|s]\dot{\xi}_\mu(s), \quad (6.8)$$

where, as usual, Greek indices run over 0, 1, 2, 3 but Latin indices run over only the 3 spatial direction. We can then rewrite the above equation (6.6) for $\alpha = 0$ as:

$$\partial_i \mathcal{E}^i(x) = \sqrt{\frac{2}{3}}\pi g \int d\tau I(\tau) \delta(x - Y(\tau)), \quad (6.9)$$

with:

$$\mathcal{E}_i(x) = \frac{1}{N} \int \delta\xi ds \epsilon_{ijk} H^{\mu jk}[\xi|s] \dot{\xi}_\mu(s) \dot{\xi}(s)^{-2} \delta(x - \xi(s)). \quad (6.10)$$

We recall next that since by (6.5), the exterior derivative of the 3-form H vanishes except on the world-line $Y_\mu(\tau)$ of the colour electric charge, the Poincaré lemma implies that $H_{\mu\nu\rho}[\xi|s]$ can be expressed in terms of a ‘dual potential’ $T_{\mu\nu}[\xi|s]$ for any point not on $Y_\mu(\tau)$. On substituting then (3.28) into (6.10), one obtains:

$$\mathcal{E}_i(x) = \epsilon_{ijk} \{ \partial^j \mathcal{T}^k(x) - \partial^k \mathcal{T}^j(x) \}, \quad (6.11)$$

with:

$$\mathcal{T}_i(x) = \frac{1}{N} \int \delta\xi ds T_{i\mu}[\xi|s] \dot{\xi}^\mu(s) \dot{\xi}(s)^{-2} \delta(x - \xi(s)), \quad (6.12)$$

where we have used the fact that the loop quantity $T_{\mu\nu}[\xi|s]$ has zero derivative in the direction along the loop. The Poincaré lemma, however, implies only the local existence of $T_{\mu\nu}[\xi|s]$ and hence also $\mathcal{T}_i(x)$, but not necessarily their global existence. Indeed, in view of the colour electric charge on $Y^\mu(\tau)$, it is clear that $\mathcal{T}_i(x)$ must be singular somewhere on any 2-sphere surrounding the colour electric charge, for if it were not so, then by cutting the sphere into two halves and applying Stokes’ theorem to respectively the northern and southern hemispheres, we would obtain:

$$\int d^3x \partial^i \mathcal{E}_i(x) = \oint \mathcal{E}_i(x) d\sigma^i = \oint_+ \mathcal{T}_i(x) dx^i + \oint_- \mathcal{T}_i(x) dx^i, \quad (6.13)$$

where the 2 resulting line integrals are both taken along the equator but in opposite directions so that their sum must vanish. On the other hand, we have from (6.9) that the integral in (6.13) must take the value $\sqrt{\frac{2}{3}}\pi g I(\tau)$ for any 2-sphere surrounding $Y(\tau)$. Hence, we have a contradiction unless \mathcal{T}_i is singular.

However, this singularity in $\mathcal{T}_i(x)$, like the Dirac string, is a mere gauge artifact, and can be avoided by patching. Introducing then two patches to cover the 2-sphere under consideration, say for example the northern and southern hemispheres overlapping along the equator, and defining for each patch a corresponding

$\mathcal{T}_i^{(N)}(x)$ or $\mathcal{T}_i^{(S)}$, then within its own patch, $\mathcal{T}_i^{(N)}$ or $\mathcal{T}_i^{(S)}$ need no longer be singular, but in the overlap region along the equator, they will have to satisfy the patching condition:

$$\oint_{eq.} \{\mathcal{T}_i^{(N)}(x) - \mathcal{T}_i^{(S)}(x)\} dx^i = \sqrt{\frac{2}{3}} \pi g I(\tau), \quad (6.14)$$

which is the same as (6.9) or equivalent to the Yang-Mills equation (6.5) defining the colour electric charge.

Our present object then is to show that the condition (6.14) is equivalent to the statement that there is an $SO(3)$ monopole of $\tilde{A}_\mu(x)$ inside the 2-sphere under consideration - i.e. a monopole defined as before as a topological obstruction via the construct detailed in Section 2, but with $\tilde{A}_\mu(x)$ as the connection. As is well-known from the work of Wu and Yang, this latter statement is the same as saying that $\tilde{A}_\mu(x)$ needs to be patched, and if we choose to adopt the same two patches as for $\mathcal{T}_i(x)$ above, then the ‘dual potential’ $\tilde{A}_\mu(x)$ in respectively the northern and southern hemispheres will have to satisfy the patching condition:

$$\tilde{A}_\mu^{(S)}(x) - S(x) \tilde{A}_\mu^{(N)}(x) S^{-1}(x) = -\frac{i}{g} \partial_\mu S(x) S^{-1}(x), \quad (6.15)$$

where $S(x)$, the patching function, has to wind once around the gauge group $SO(3)$ (i.e. half-way round $SU(2)$) as x winds once around the equator. Equivalently, we have:

$$P \exp \oint_{eq.} \{\partial_i S(x) S^{-1}(x)\} dx^i = -1. \quad (6.16)$$

Thus, if we are able to show that the patching condition (6.16) for $S(x)$ is equivalent to that for $\mathcal{T}_i(x)$, namely (6.14), we will have shown that a colour electric charge is indeed a monopole of $\tilde{A}_\mu(x)$, and vice versa. Moreover, since both the conditions (6.14) and (6.16) are invariant under \tilde{U} -transformations (the former because under a \tilde{U} -transformation $\mathcal{T}_i(x)$ changes only by a total derivative whose integral over a closed loop vanishes), it will be sufficient to demonstrate their equivalence in some particular \tilde{U} -gauge choice.

Now by its definition (5.15) above, one sees that $\tilde{A}_i(x)$ is related to $\mathcal{T}_i(x)$ by:

$$\omega^{-1}(x) \tilde{A}_i(x) \omega(x) = -2\sqrt{6} \mathcal{T}_i(x), \quad (6.17)$$

which holds for the patched quantities each in the patch in which it is defined. We may thus write:

$$\omega^{(N)-1}(x) \tilde{A}_i^{(N)}(x) \omega^{(N)}(x) - \omega^{(S)-1}(x) \tilde{A}_i^{(S)}(x) \omega^{(S)}(x) = -2\sqrt{6} \{\mathcal{T}_i^{(N)}(x) - \mathcal{T}_i^{(S)}(x)\}, \quad (6.18)$$

where we recall that $\omega(x)$ is by definition the rotation matrix which transforms from the local U -gauge to the local \tilde{U} -gauge. Hence, $\omega^{(S)}(x) \omega^{(N)-1}(x)$ is the matrix which transforms from the local northern \tilde{U} -gauge to the local southern \tilde{U} -gauge, or in other words the patching function is given by:

$$S(x) = \omega^{(S)}(x) \omega^{(N)-1}(x). \quad (6.19)$$

Thus, from (6.18), we have:

$$\omega^{(S)-1}(x)\{\tilde{A}_i^{(S)}(x) - S(x)\tilde{A}_i^{(N)}(x)S^{-1}(x)\}\omega^{(S)}(x) = 2\sqrt{6}\{\mathcal{T}_i^{(N)}(x) - \mathcal{T}_i^{(S)}(x)\}, \quad (6.20)$$

or by the patching condition (6.15) for $\tilde{A}_i(x)$:

$$\omega^{(S)-1}(x)\{\partial_i S(x) S^{-1}(x)\}\omega^{(S)}(x) = 2\sqrt{6}i\tilde{g}\{\mathcal{T}_i^{(N)}(x) - \mathcal{T}_i^{(S)}(x)\}. \quad (6.21)$$

Next, we go to the special \tilde{U} -gauge where both $\omega^{(N)}(x)$ and $\omega^{(S)}(x)$ point in the same fixed ‘colour’ direction. This is possible since we can make independent gauge rotations in the two patches. In that case, $S(x)$ will also point in the same fixed ‘colour’ direction and commute with $\omega^{(S)}(x)$, giving by (6.21) above:

$$\partial_i S(x) S^{-1}(x) = 2\sqrt{6}i\tilde{g}\{\mathcal{T}_i^{(N)}(x) - \mathcal{T}_i^{(S)}(x)\}, \quad (6.22)$$

which shows that in this gauge the right-hand side also points in the same fixed ‘colour’ direction. In that case, since the exponents at different points commute, we can evaluate the holonomy on the left-hand side of (6.16) without accounting for the path-ordering, and write just:

$$P \exp \oint_{eq.} \{\partial_i S(x) S^{-1}(x)\} dx^i = \exp 2\sqrt{6}i\tilde{g} \oint_{eq.} \{\mathcal{T}_i^{(N)}(x) - \mathcal{T}_i^{(S)}(x)\} dx^i. \quad (6.23)$$

Recalling then the Dirac quantisation condition relating the unit (colour) electric and magnetic charges, which for the $su(2)$ theory reads as: [25]

$$g\tilde{g} = \frac{1}{4}, \quad (6.24)$$

we see that the conditions (6.14) and (6.16) are indeed equivalent as we wanted.

Given now that a colour (electric) charge as usually defined in the Yang-Mills theory has been shown to be equivalent to a monopole of the dual connection $\tilde{A}_\mu(x)$, we can then in principle apply the same Wu-Yang criterion that we have used in all previous cases for monopoles to derive the equations of motion for a colour electric charge. All we have to do is to take the free action of the field-particle system expressed in terms of the \tilde{U} -gauge invariant field variable, namely $H_{\mu\nu\rho}[\xi|s]$, and extremise with respect to this variable and the particle world-line or wave function as the case may be, but subject to the defining constraint of the monopole. This is a very different approach to the conventional one for a colour electric charge as outlined at the beginning of this section, and although the analogy to the abelian theory suggests that the resultant equations may be the same as the standard equations, there is *a priori* no guarantee at all that this will indeed be the case. Nevertheless, as we shall see, the answer does turn out to be exactly as expected for both a classical and a Dirac point particle carrying a colour electric charge.

For a classical point charge, the free action is:

$$\tilde{\mathcal{A}}^0 = \tilde{\mathcal{A}}_F^0 - m \int d\tau, \quad (6.25)$$

with \tilde{A}_F^0 as given in (4.1). The same arguments as those outlined in Section 5 for the colour magnetic charge will lead to the conclusion that a colour electric charge in $SO(3)$ theory, when considered as a monopole of $\tilde{A}_\mu(x)$, can take the values \pm , where $-$ corresponds to a monopole and $+$ no monopole. Further, the quantity $\delta_\mu(s)\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}[\xi|s]$, which is the equivalent of $G_{\mu\nu}[\xi|s]$ for the colour magnetic charge, will take a value ι satisfying (cf (5.1)):

$$\exp i\pi\iota = -1, \quad (6.26)$$

at the position of the colour electric charge but is zero elsewhere. Hence, in analogy to (5.2) and (5.3), one can write the defining constraint for the classical colour electric charge as:

$$\delta_\mu(s)\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}[\xi|s] = -4\pi J[\xi|s] \quad (6.27)$$

with:

$$J[\xi|s] = \sqrt{6}g \int d\tau \iota[\xi|s] \frac{dY^\mu(\tau)}{d\tau} \dot{\xi}_\mu(s) \delta(\xi(s) - Y(\tau)). \quad (6.28)$$

Extremising then (6.25) under the constraint (6.27) with respect to $H_{\mu\nu\rho}[\xi|s]$ gives the equation (4.7), which, as shown in Section 4, is equivalent to the Yang-Mills condition that the gauge potential $A_\mu(x)$ exists. Extremising with respect to $Y^\mu(\tau)$, on the other hand, gives:

$$\begin{aligned} m \frac{d^2 Y^\mu(\tau)}{d\tau^2} &= 4\sqrt{6}\pi g \int \delta\xi ds \left[\delta_\sigma(s) \text{Tr}\{L[\xi|s]\iota[\xi|s]\} \dot{\xi}^\mu(s) \frac{dY^\sigma(\tau)}{d\tau} \right. \\ &\quad \left. - \delta^\mu(s) \text{Tr}\{L[\xi|s]\iota[\xi|s]\} \dot{\xi}_\sigma(s) \frac{dY^\sigma(\tau)}{d\tau} \right] \delta(\xi(s) - Y(\tau)). \end{aligned} \quad (6.29)$$

The Lagrange multiplier $L[\xi|s]$ can be eliminated from the equation (6.29) using the field equation (4.7) and the fact that:

$$\delta_\alpha(s)\iota[\xi|s] = 0, \quad (6.30)$$

which is a consequence of the constraint (6.5) and the fact that $H_{\mu\nu\rho}[\xi|s]$ has zero longitudinal loop derivative. The result is:

$$\begin{aligned} m \frac{d^2 Y^\mu(\tau)}{d\tau^2} &= \sqrt{\frac{2}{3}} \frac{g}{N} \int \delta\xi ds \left[\epsilon_{\alpha\nu\rho\sigma} \text{Tr}\{H^{\alpha\nu\rho}[\xi|s]\iota[\xi|s]\} \dot{\xi}^\mu(s) \dot{\xi}(s)^{-2} \frac{dY^\sigma(\tau)}{d\tau} \right. \\ &\quad \left. - \epsilon^{\alpha\nu\rho\mu} \text{Tr}\{H_{\alpha\nu\rho}[\xi|s]\iota[\xi|s]\} \dot{\xi}_\sigma(s) \frac{dY^\sigma(\tau)}{d\tau} \right] \delta(\xi(s) - Y(\tau)). \end{aligned} \quad (6.31)$$

By means then of the ‘duality relation’ (3.27) giving $H_{\mu\nu\rho}[\xi|s]$ in terms of $F_\mu[\xi|s]$, one obtains, on substituting for $F_\mu[\xi|s]$ its expression (3.13) in terms of ordinary local variables, exactly the Wong equation (6.4) in place of (6.31). Similarly, the defining constraint (6.5) itself reduces to the other Wong equation (5.10). We have thus, as anticipated, rederived the Wong equations using an entirely

different approach by treating the colour electric charge as a monopole of the dual connection $\tilde{A}_\mu(x)$ and applying to it the Wu-Yang criterion. An added interest is that the equations are now derived from an action principle, which, as we recall, was not available in the conventional approach for deriving the Wong equations.

Next, for the Dirac particle, we start with the free action:

$$\tilde{\mathcal{A}}^0 = \tilde{\mathcal{A}}_F^0 + \int d^4x \bar{\psi}(x)(i\partial_\mu \gamma^\mu - m)\psi(x), \quad (6.32)$$

with $\tilde{\mathcal{A}}_F^0$ as given in (4.1) under the defining constraint (6.5) of the monopole with:

$$J[\xi|s] = \sqrt{6}g[\bar{\psi}(\xi(s))t_i\gamma^\mu\psi(\xi(s))]t^i\dot{\xi}_\mu(s), \quad (6.33)$$

where we have again just replaced the classical current in (6.28) by the quantum current. On extremising:

$$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}^0 + \int \delta\xi ds \text{Tr}\{L[\xi|s][\delta_\mu(s)\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}[\xi|s] + 4\pi J[\xi|s]]\} \quad (6.34)$$

with respect to $H_{\mu\nu\rho}[\xi|s]$ we obtain the equation (4.7), and with respect to $\psi(x)$, the standard Yang-Mills equation (6.3) for $\psi(x)$ with $A_\mu(x)$ given in terms of the Lagrange multiplier $L[\xi|s]$ as per (4.11). In other words, in as much as (4.7) has already been shown in Section 4 to be equivalent to the usual Yang-Mills field equation with $A_\mu(x)$ as the gauge potential, we have here rederived exactly the standard Yang-Mills theory by means of the Wu-Yang criterion treating the colour (electric) charge as a monopole, and without recourse to either an interaction term in the action or an appeal to the minimal coupling hypothesis.

Needless to say, these equations, being just the standard Yang-Mills equations, are invariant under the usual U gauge transformations. We have thus completed on the right-hand side of Chart IV the sought-for analogy with the left-hand side of the same Chart, and also with the right-hand side of Chart II.

7 Remarks

In the four preceding sections we have shown how, by introducing a generalised duality relationship for Yang-Mills fields, one can reproduce nonabelian analogues for all the dual properties of electromagnetism listed in Chart I and II. Indeed, the analogy can be made to appear even closer by a few changes in notation. Thus, by defining:

$$E_\mu[\xi|s] = \Phi_\xi(s, 0)F_\mu[\xi|s]\Phi_\xi^{-1}(s, 0), \quad (7.1)$$

and noting that the constraints signifying the absence of colour electric and magnetic charges in the pure Yang-Mills theory can be written respectively as:

$$\delta_\mu E^\mu[\xi|s] = 0, \quad (7.2)$$

and:

$$\delta_\rho H^{\mu\nu\rho}[\xi|s] = 0, \quad (7.3)$$

one can recast Chart III into the form shown as Chart V, the analogy of which to Chart I for pure electromagnetism is very obvious. A similar change of notation will also reveal a closer analogy of Chart IV with Chart II.

However, the more symmetric-looking Chart V misses one important piece of information that Chart III contains, namely the fact that the field can actually be described in terms of just the local potential $A_\mu(x)$, which is of course essential for the theory to qualify as a local gauge theory. This important difference between the charts also highlights a very significant lack of symmetry between the left- and right-hand sides in each of the charts for the Yang-mills case, namely between the ‘direct’ and ‘dual’ formulations of the theory. Although in both formulations, one has been able to recover local quantities $A_\mu(x)$ and $\tilde{A}_\mu(x)$ which act as parallel phase transports for the wave functions of respectively colour electric and colour magnetic charges, only in the direct formulation has it been shown that the field can in principle be described just in terms of the local quantity $A_\mu(x)$. In other words, the quantity $\tilde{A}_\mu(x)$ in the dual formulation has acquired so far only the significance of a local phase transport in parallel to $A_\mu(x)$, but not as yet, like $A_\mu(x)$, also the significance of a local potential from which all field quantities can be derived. As far as we can see at present, it does not seem excluded that in the dual formulation also, all field quantities may eventually be shown to be expressible in terms of $\tilde{A}_\mu(x)$, but we have not succeeded in doing so. It might even happen that when expressed in terms of $\tilde{A}_\mu(x)$, an exact dual symmetry would be restored for Yang-Mills fields through a generalised duality relationship of which we at present know only the loop space form, namely (3.27), but we have no idea at present whether this may or may not be the case.

In the absence of this knowledge, one can only assume that Yang-Mills theory is not dual symmetric, or that unlike electromagnetism, (colour) electric and magnetic charges are governed by different dynamics as regards their interactions with the field. For the colour electric charge, one sees that, in spite of the very different treatment reported above from the conventional, the dynamics turns out to be exactly the standard local theory of a Yang-Mills colour source, which is well-known. For the colour magnetic charge, on the other hand, we have at present only the ‘dual’ formulation in terms of loop variables which is unfortunately somewhat unwieldy. Nevertheless, the equations of motion have been derived, as shown above, and with some more persistence, the formulation can be cast into a Lagrangian form for Feynman diagrams of colour magnetic charges to be evaluated. Some initial exploration in this direction has been made and will be reported elsewhere.[16] Of course, the question of whether colour magnetic charges will have any physical significance can only be answered when more is known about their dynamics.

One novel feature of the above treatment for both (colour) electric and (colour) magnetic charges is that their dynamics has been derived throughout using the same universal Wu-Yang criterion for monopoles. For colour electric charges, the

application of this criterion is made possible by the new result derived in Section 6 that a colour (electric) charge defined as usual as a source of the Yang-Mills field turns out to be also a monopole of the dual phase transport $\tilde{A}_\mu(x)$. In this treatment, the interaction between the charge and the field arose purely as the consequence of the definition of the charge as a topological obstruction in the field. It gives a unique answer which coincides exactly with the conventional form of the interaction deduced from a very different line of argument in all cases where the latter exist, namely the Lorentz and Dirac equations for ordinary electric and magnetic charges, and the Wong and Yang-Mills equations for the colour electric charge. The result is valid in principle for all Yang-Mills theory (although we call the field ‘colour’ here for convenience) which means that all known charges can be interpreted as monopoles and all their physical forces (except gravitation) have now been derived by this method. The approach has arguably some aesthetic appeal versus the conventional in that monopoles occur ‘naturally’ in gauge theories as topological obstructions while sources do not seem to have any obvious geometrical significance, and that their interactions then follow from the Wu-Yang criterion uniquely without apparently any appeal to the minimal coupling hypothesis.

For the nonabelian theory, it is interesting to note that a colour charge (whether electric or magnetic), when considered as a monopole, is by definition actually labelled only by a homotopy class of the gauge group and has thus initially no colour direction, i.e. direction in internal symmetry space. It was in writing down the defining constraint as an equation in the gauge Lie algebra, namely either (5.2), to formulate the dynamics that we first assign a colour direction to the charge. It follows then that this direction cannot have any real physical significance, or that the dynamics must be invariant under both U and \tilde{U} gauge transformations, as was indeed found to be the case. Indeed, for the classical point charge, it is possible to write down the defining constraint as an equation in the gauge group without giving the charge a colour direction, so that the dynamics can be at least stated, though not perhaps implemented, entirely in terms of gauge invariants without introducing a concept of gauge. However, we do not know what the case is for a quantum charge.

As already stressed in the introduction, the results presented in this paper concern ordinary Yang-Mills theory in strictly 4 space-time dimensions, only stated in the somewhat unconventional loop-space language. This loop-space treatment we believe to be appropriate for the reasons already stated, but though powerful, the method has certain drawbacks requiring sometimes quite delicate handling, for example, in having to take limits and derivatives in a prescribed order which, we think, have to do with the uncertainties in the functional approach itself. In this work, we have dealt with the individual cases as they arose, but have often felt the lack of a general calculus for handling complex loop space operations.

Finally, it will be interesting to ask in what manner our results above are related to the recent work of Seiberg, Witten and co-workers. Given that the ques-

tion addressed is in both cases essentially the same, it seems that a relationship must eventually exist, but since the two parallel approaches to monopoles have been developed independently for some years, it is not immediately clear how they are actually related. However, our result here may have a complementary value with respect to the other more ambitious program in that the relationship between the dual field quantities, albeit rather complicated, is here explicitly given for Yang-Mills fields, a relationship which, according to Seiberg, is being urgently sought in their approach.

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